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Computational analysis of Ramsey-type theorems

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Résumé / Abstract

Version française

Dans cette thèse, nous étudions le contenu calculatoire du théorème de Ramsey ainsi que d'autres énoncés associés. Pour ce faire, nous utilisons différentes notions de réductibilité et des outils provenant de la théorie de la calculabilité et des mathématiques à rebours. Nous utilisons des techniques récentes pour établir différents nouveaux résultats de séparation entre des énoncés de Ramsey, en fonction du nombre de couleurs et d'instances. Nous obtenons également une équivalence entre des énoncés combinatoires sur les arbres et une variation du théorème de Ramsey, nous permettant ainsi d'unifier des résultats précédemment connus, tout en en fournissant de nouveaux.

English version

In this thesis, we investigate the computational content of Ramsey's theorem and other related statements. To do so, we use various notions of reducibility and tools provided by the frameworks of computability theory and reverse mathematics. We use recent techniques to establish several new separation results between Ramsey statements, depending on the number of colors and instances. We also obtain an equivalence between some combinatorial statements on trees and a variation of Ramsey's theorem, thus unifying previously known results, while providing new ones.

Mots clés / Keywords

- logique, calculabilité, mathématiques à rebours, théorème de Ramsey
- logic, computability theory, reverse mathematics, Ramsey's theorem

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Résumé détaillé de la thèse

Ce document est un rapport de thèse issu de trois années de recherche au sein du LACL (Laboratoire d'Algorithmique, Complexité et Logique) à l'université Paris-Est Créteil (UPEC), sous la direction de Julien Cervelle et Ludovic Levy Patey. Il traite du contenu calculatoire des énoncés mathématiques liés à un théorème de combinatoire connu sous le nom de théorème de Ramsey. Les différentes études sont réalisées en utilisant les outils de la calculabilité et des mathématiques à rebours. Nous commençons par présenter brièvement tout cela.

Théorie de la calculabilité

La théorie de la calculabilité est une branche de la logique mathématique apparue au début du $20^{\text{ème}}$ siècle suite à la crise des fondements, elle étudie les limites de ce qui est calculable par algorithmes. Elle a tout d'abord permit de définir ce qu'est un algorithme et d'établir qu'il s'agit d'une notion robuste, c'est-à-dire que l'on possède plusieurs modèles mathématiques qui semblent capturer cette notion, et dont la puissance est la même. Ce fait empirique est connu sous le nom de "thèse de Church-Turing".

Parmi les modèles équivalents, les plus connus sont les machines de Turing, le λ-calcul, et les fonctions μ-récursives. Une fonction $f : \mathbb{N} \to \mathbb{N}$ est alors dite "calculable" lorsqu'il existe un algorithme qui, pour toute entrée n, donne en sortie $f(n)$. Cette définition peut s'adapter pour considérer plusieurs entiers en entrée. En particulier un ensemble d'entiers A est dit calculable s'il existe un algorithme qui, pour tout entier n en entrée, donne 1 en sortie si $n \in A$ et 0 si $n \notin A$. Rapidement, par des arguments de diagonalisation, l'existence d'ensemble non-calculables a été établit, et un exemple important finit par émerger, il s'agit du "problème de l'arrêt". Ce dernier peut être encodé par un ensemble et consiste à savoir si, pour un algorithme et une entrée donnés, nous sommes capables de dire si cet algorithme finira par s'arrêter sur cette entrée. À partir de cela, c'est toute une hiérarchie de complexité des ensembles d'entiers qui a été mis à jour : la hiérarchie de Turing. Cette dernière repose sur la réduction Turing, notée $\leqslant_T$, définit par $A \leqslant_T B$, lorsqu'il existe un algorithme qui peut calculer A si on lui donne accès à l'ensemble B (i.e. sa fonction caractéristique). Son étude a suscité de nombreux travaux de la part des chercheurs, ce qui a considérablement fait avancer la calculabilité par la mise au point de notions, de théorèmes, et plus généralement de techniques de construction avancées, dans le but d'obtenir des ensembles vérifiant des propriétés bien précises. Parmi ces méthodes les plus importantes sont le forcing et la méthode de priorité (à blessures finies ou infinies). Néanmoins, sa compréhension est encore incomplète et il reste à ce jour des questions ouvertes importantes.

Mathématiques à rebours

Les mathématiques à rebours, en anglais reverse mathematics, sont une branche de la logique mathématique, fondée en 1974 par Harvey Friedman. Le but originel du domaine était de chercher à savoir quels sont les axiomes les plus faibles permettant de prouver un théorème donné, d'où le nom. De manière plus générale, les mathématiques à rebours donnent un cadre formel dans lequel les outils provenant de la théorie de la calculabilité et la théorie de la preuve peuvent être utilisés pour mettre à jour l'aspect constructif des théorèmes, et comment ces derniers interagissent les uns avec les autres. En outre, il existe des notions formelles correspondant au fait qu'un théorème est prouvable à partir d'un autre. Ces objectifs amènent très souvent à établir de nouvelles preuves, axiomatiquement plus simples, pour des théorèmes connus.

Le cadre formel des mathématiques à rebours est celui de logique du second ordre. C'est-à-dire que les formules logiques considérées peuvent quantifier sur des prédicats. Plus précisément on s'intéresse à certains sous-systèmes axiomatiques de l'arithmétique du second ordre. Ce choix provient du fait que beaucoup d'objets mathématiques peuvent être encodés par des entiers ou ensembles d'entiers, ainsi il devient possible d'exploiter les puissants outils développés en calculabilité. A titre d'exemple, les fonctions continues de $\mathbb{R}$ dans $\mathbb{R}$ peuvent être encodées par des ensembles d'entiers. Pour cela il faut utiliser le fait que $\mathbb{R}$ possède une base dénombrable, et que la continuité peut s'exprimer en parlant uniquement de la préimage des éléments de cette base.

Dans ce cadre, les théorèmes peuvent alors s'exprimer par des énoncés du second ordre, de la forme $\forall X, (\Phi(X) \implies \exists Y, \Psi(X,Y))$, où Φ et Ψ sont des formules arithmétiques, que l'on peut comprendre de la manière suivante : étant donné un problème P, si X est une instance de P, alors Y est une solution à l'instance X de P. Un exemple bien connu en mathématiques à rebours est le lemme faible de König, noté WKL, qui dit que pour tout arbre binaire infini possède un chemin infini. Une instance de WKL est un arbre binaire infini T, et une solution est un chemin infini dans T. Représenter les théorèmes de cette manière permet de les manipuler et de définir plusieurs notions de “réduction”, chacune précisant l'idée qu'un théorème est plus fort qu'un autre. L'une d'entre elles est la réduction calculatoire, qui dit la chose suivante. Un problème P se réduit calculatoirement à Q, noté $\mathsf{P} \leqslant_c \mathsf{Q}$, si, toute instance I de P calcule une instance $\widehat{I}$ de Q telle que, pour toute solution $\widehat{S}$ de $\widehat{I}$, on a qu'une solution à I peut être calculée à partir des deux ensembles I et $\widehat{S}$.

Les mathématiques à rebours définissent cinq sous-systèmes axiomatiques de l'arithmétique du second-ordre, appelés “Big Five”. Le plus faible d'entre eux s'appelle RCA_0, et il sert de base pour les raisonnements. Il correspond globalement aux mathématiques calculables, et permet de dire qu'un théorème P est plus puissant qu'un théorème Q lorsque $\mathsf{RCA}_0 + \{\mathsf{P}\} \vdash \mathsf{Q}$. Les quatre autres sous-systèmes sont des extensions de RCA_0, linéairement (et strictement) ordonnés par la relation que nous venons de décrire. Dans cette thèse nous ne nous intéresseront qu'aux trois premiers (par ordre de puissance), à savoir RCA_0, WKL_0 et ACA_0. Une classification empirique étonnante est très vite apparue concernant les Big Five : étant donné un théorème P des mathématiques “classiques” (que l'on peut exprimer en arithmétique du second ordre, typiquement les théorème des valeurs intermédiaires), soit il est prouvable dans RCA_0, soit $\mathsf{RCA}_0 + \{\mathsf{P}\}$ est équivalent à un des quatre autres sous-systèmes.

C'est alors que le théorème de Ramsey entre en jeu, car il est le premier exemple d'un théorème échappant à ce phénomène. Ceci explique l'intérêt particulier que lui porte la communauté des mathématiques à rebours qui, depuis près de 50 ans maintenant, élabore et raffine des techniques complexes afin de mieux le comprendre.

Le théorème de Ramsey

Étant donné un ensemble $X \subseteq \mathbb{N}$ et un entier $n \in \mathbb{N}$, on désigne par $[X]^n$ la collection des sous-ensembles de X de cardinal n. Pour tout ensemble $X \subseteq \mathbb{N}$ et $k \in \mathbb{N}$, un k-coloriage de X est une fonction de X dans $\{0, \dots, k-1\}$. Le théorème de Ramsey a été découvert en 1928 par le mathématicien Frank Ramsey. Pour deux entiers n et k donnés, on le note RT^n_k, et il affirme que pour tout k-coloriage f de $[\mathbb{N}]^n$, il existe un sous-ensemble infini $H \subseteq \mathbb{N}$ tel que card $\big(f\big([H]^n\big)\big) = 1$. Un tel ensemble est dit f-homogène. Lorsque $n = 1$ on retrouve une variante du principe des tiroirs : une partition finie de $\mathbb{N}$ contient nécessairement un ensemble infini. En revanche, à partir de $n = 2$ il devient bien plus complexe, c'est d'ailleurs pour cette valeur que le théorème échappe aux Big Five. Plusieurs décennies de recherche et l'étude de divers théorèmes proches de celui de Ramsey ont été nécessaires afin de prouver ce résultat. Ces variations du théorème de Ramsey sont dit "de type Ramsey".

Plus concrètement, il a été montré que RT^n_k et RT^n_ℓ sont équivalents au-dessus de RCA_0, pour tout $n \in \mathbb{N}$ et tout $k, \ell \geqslant 2$. Ainsi, pour chaque n, il suffit de s'intéresser à $k = 2$. Pour $n = 1$, on peut prouver RT^1_2 dans RCA_0. Pour $n \geqslant 3$, il a été prouvé que $\mathsf{RCA}_0 + \{\mathsf{RT}^n_k\}$ est équivalent à ACA_0. Le cas $n = 2$ a été beaucoup plus difficile à déterminer, mais il a été montré que $\mathsf{RCA}_0 + \{\mathsf{RT}^2_2\}$ est strictement compris entre ACA_0 et RCA_0, tout en étant incomparable à WKL_0.

La thèse

La présente thèse se place donc dans la lignée des travaux décrits ci-dessus. Elle s'articule en quatre chapitres, le premier étant une introduction aux domaines et résultats connus, tandis que les trois autres contiennent des contributions nouvelles concernant l'analyse de la puissance calculatoire de divers théorèmes de type Ramsey, et sont assez indépendants. Nous résumons désormais le contenu de chaque chapitre.

Chapitre 1

Le chapitre 1 introduit le bagage de connaissances nécessaires à la compréhension des autres chapitres. Bien qu'une certaine familiarité soit attendue de la part du lecteur, nous passons en revue les bases de la calculabilité et des mathématiques à rebours. On aborde notamment la hierarchie arithmétique, l'hyperimmunité, les classe Π^0_1 et théorèmes de base, les degrés PA, ainsi que le forcing. Nous proposons également un petit résumé historique concernant le théorème de Ramsey et ses résultats, du point de vue des mathématiques à rebours. Ceci afin de donner une appréciation plus substantielle et globale de son étude, notamment en présentant une partie de ses articles et contributeurs.

Chapitre 2

Le chapitre 2 explore des variantes du théorème de Ramsey pour les paires (RT^2_2), où le coloriage en entrée et l'ensemble infini en sortie peuvent être restreints. Par exemple, pour un coloriage $f : [\mathbb{N}]^2 \to 2$, on peut demander à l'ensemble solution de ne pas être f-homogène, mais seulement f-transitif, i.e. $\forall i < 2, \forall x < y < z, f(x,y) = f(y,z) = i \implies f(x,z) = i$. Ces théorèmes nous amènent à concentrer notre étude sur une variante de l'énoncé CAC dans le cas particulier où l'ordre considéré est la relation de prédécesseur dans un arbre. Cet énoncé, bien connu en mathématiques à rebours, affirme que tout ordre infini partiel possède soit une chaine infinie, soit une antichaine infinie. Sa variante sur les arbres s'est avérée avoir de multiples caractérisations, ce qui en fait une notion robuste. Parmi ses énoncés équivalents on note une variante de RT^2_2 appelée SHER, ainsi que l'énoncé TAC, déjà étudié de manière indépendante par Conidis, et qui porte sur les arbres binaires complètement branchants, c'est-à-dire les arbres tels que $\forall i < 2, \forall \sigma \in T, (\sigma \cdot i \in T \implies \sigma \cdot (1-i) \in T)$. L'énoncé affirme qu'un arbre c.e. binaire infini qui est complètement branchant contient une antichaine infinie. Ainsi, plusieurs théorèmes établis indépendamment (notamment par Conidis et Dorais) ont été obtenus ou améliorés, dans un cadre plus unifié. Les résultats de ce chapitre ont été acceptés pour publication dans le Journal of Symbolic Logic [CGP23].

Chapitre 3

Le chapitre 3 cherche à établir le résultat de séparation suivant : $\forall n, \mathsf{SRT}^n_3 \not\leq_c (\mathsf{RT}^n_2)^*$, où SRT^n_k désigne le théorème de Ramsey où les coloriages $f : [\mathbb{N}]^n \to k$ considérés sont stables, i.e. ils vérifient $\forall \vec{x} \in [\mathbb{N}]^{n-1}, \exists i < k, \exists s, \forall y > s, f(\vec{x}, y) = i$. Autrement dit, on établit qu'un produit arbitraire fini du RT^n_2 ne peut pas calculatoirement résoudre un unique SRT^n_3. Il s'agit d'une amélioration d'un théorème établit par Liu, à savoir $\mathsf{SRT}^2_3 \not\leq_c (\mathsf{SRT}^2_2)^*$.

Pour arriver à ce résultat, nous nous basons sur une méthodologie récente et technique, mise en place par Liu, afin de prouver des théorèmes de base et de préservation dans le cadre du problème "cross-constrait". Une étape clé consiste à prouver qu'une variante de l'hyperimmunité, nommée Γ-hyperimmunité, est préservée par l'énoncé COH.

Chapitre 4

Ce chapitre, plus court que les précédents, établit un résultat de séparation pour une réduction dite “omnisciente forte”. On désigne par $\mathsf{RT}^n_{k,q}$ la variante du théorème de Ramsey où l’ensemble solution doit utiliser moins de q couleurs (au lieu d’une seule dans le cas habituel). On prouve dans ce chapitre que $\mathsf{RT}^1_{q+1,q} \not\leq_{soc} \mathsf{SRT}^2_{<\infty,q+1}$. Pour arriver à cela, on se base sur une construction par forcing de Dzhafarov, Patey, Solomon, et Westrick.

Short thesis summary

This document is a thesis report resulting from three years of research at LACL (Laboratoire d’Algorithmique, Complexité et Logique), under the supervision of Julien Cervelle and Ludovic Levy Patey. It deals with the computational content of mathematical statements related to a theorem of combinatorics known as Ramsey’s theorem. The different studies are carried out by using the tools of computability theory and reverse mathematics.

Computability theory provides the necessary equipment to deal with the notions of algorithm and computation. In particular, it uncovers a whole hierarchy of complexity for sets, and supplies advanced techniques and theorems to obtain sets with certain properties.

Reverse mathematics use the framework of second-order arithmetic. The objects we are interested in must be encodable as integers or sets of integers. Theorems are then second-order formulas that are generally seen as problems with instances and solutions. More importantly, there are formal notions of “reducibility” that make precise the idea of one theorem being stronger than another.

In this framework, the goal of many researchers has been to establish a detailed picture of the relationships between known theorems, in order to better understand them. In that regard, Ramsey’s theorem has received a particular interest, as it did not fit in the “Big Five” categories under which other theorems usually fall. This thesis contributes to this classification with new theorems and by simplifying the context of some known results.

Here we briefly explain the content of each chapter.

- In Chapter 1, we introduce the background of knowledge necessary for the understanding of the other chapters. Although some familiarity is expected from the reader, we go through the basics of computability theory and reverse mathematics. We also offer a short survey on Ramsey’s theorem in reverse mathematics.

- In Chapter 2, we explore variations of Ramsey's theorem for pairs (RT^2_2), where either the input coloring or the output infinite set is restricted. We focus the study on a variation of the well-studied statement CAC, in the particular case where the order considered is the predecessor relation in a tree. This statement turned out to have multiple characterizations, making it a robust notion. In particular, one equivalent statement is a variation of RT^2_2 called SHER. Some theorems presented here are independent rediscoveries but were obtained in a more unified setting. The results of this chapter were published in the Journal of Symbolic Logic [CGP23].
- In Chapter 3, we explore computable reduction for multiple instances of Ramsey's theorem. We establish a separation result by using basis and preservation theorems for what we call the "cross-constrain problem". This is done with a recent combinatorial technique and a variation of hyperimmunity.
- Finally in Chapter 4, we establish a separation result, for strong omniscient reduction, between Ramsey's theorem for singletons and its stable counterpart for pairs, depending on the number of colors involved.

Main contributions

In this section, we highlight the main theorems uncovered during this thesis.

Chapter 2

By combining Theorem 2.2.4 and Proposition 2.6.3 + Proposition 2.6.6, we established the following equivalence between three different statements.

Theorem. *The following statements are equivalent over* RCA_0 *and computable reduction:*
(1) CAC for trees
(2) $\mathsf{TAC} + \mathsf{B}\Sigma^0_2$
(3) SHER

With Proposition 2.3.3+Lemma 2.2.11, we established that TAC has a probabilistic proof. This is an interesting property since very few theorems studied in reverse mathematics possess it.

Theorem. *The measure of the oracles computing a solution for a computable instance of* TAC *is 1.*

With Theorem 2.5.1, we provided a general statement regarding TAC that encompasses some known results.

Theorem. *Let* $(A_n)_{n\in\mathbb{N}}$ *be a uniformly* Δ^0_2 *sequence of infinite* Δ^0_2 *sets. There is a computable instance of* TAC *such that no* A_n *is a solution.*

Finally, with Corollary 2.7.9 we have an equivalence between the stable counterparts of the statements mentioned above.

Theorem. *The following are equivalent over* RCA_0*:*
(1) CAC for stable trees
(2) SHER *for stable colorings*

Chapter 3

Theorem 3.3.14 is a technical result that is essential to prove the separation theorems of this chapter.

Theorem. *Let $\mathcal{M}$ be a countable cross-constraint ideal such that $\mathcal{M} \models \mathsf{COH}$ and let $f : \mathbb{N} \to 3$ be hyperimmune relative to any element of $\mathcal{M}$, then for any $r \in \mathbb{N}$ and any $g_0, \dots, g_{r-1} : [\mathbb{N}]^2 \to 2$ in $\mathcal{M}$, there are infinite g_i-homogeneous sets G_i for every $i < r$, such that $\bigoplus_{i<r} G_i$ does not compute any infinite f-homogeneous set.*

Many results of this chapter are basis and preservation theorems related to the cross-constraint problem.

Firstly, Theorem 3.4.19 is the equivalent of the non-Σ^0_1 basis theorem.

Theorem (Cross-constraint preservation of non-Σ^0_1 definitions)**.** *Let C be a non-Σ^0_1 set. Any computable instance T of CC, has a solution $(X^i, Y^i)_{i<2}$ such that C is not Σ^0_1 relative to $(X^0, Y^0) \oplus (X^1, Y^1)$.*

Secondly, Theorem 3.4.23 is the equivalent of the low basis theorem.

Theorem (Cross-constraint low basis)**.** *Any left-full computable instance T of CC, has a solution $(X^i, Y^i)_{i<2}$ such that $(X^0, Y^0) \oplus (X^1, Y^1)$ is low.*

And thirdly, Theorem 3.5.16 states that the notion of Γ-hyperimmunity is preserved by COH.

Theorem. *Let $g \in 3^{\mathbb{N}}$ be a Γ-hyperimmune function and $R_0, R_1, \dots$ be a uniformly computable sequence of sets. Then there is an $\vec{R}$-cohesive set G such that g is Γ-hyperimmune relative to G.*

The results established in this chapter ultimately lead to the separation stated in Theorem 3.5.21.

Theorem. *For every $n \geqslant 2$, $\mathsf{SRT}^n_3 \not\leqslant_c (\mathsf{RT}^n_2)^*$*

Chapter 4

The main result of this chapter is Theorem 4.0.2, which establishes the following separation.

Theorem. *For all $q \geqslant 2$, we have* $\mathsf{RT}^1_{q+1,q} \not\leqslant_{soc} \mathsf{SRT}^2_{<\infty,q+1}$

CHAPTER 1

INTRODUCTION

This thesis deals with the computational content of mathematical statements related to a theorem of combinatorics known as Ramsey's theorem. To do so, it uses the framework and tools of computability theory and reverse mathematics. In this first chapter we give the relevant prerequisites necessary to the understanding of the other chapters. Fundamental definitions and results of computability theory and reverse mathematics are presented. A particular focus is given to the study of Ramsey's theorem in reverse mathematics, since it is at the center of the different contributions of this thesis. Nonetheless it is recommended to have some familiarity with these topics before reading past the introduction. For a more in-depth introduction, the author recommends [Sim09], [Hir15], [Soa16], [DM22] and [MP22].

1.1 Notation

Before diving into the different subjects, we define some basic notation that will be used throughout the entire document.

1.1.1 Set theoretic notation

Given a set A, its **power set** is denoted by $\mathfrak{P}(A)$. A binary relation $\bowtie$ on a set A is **well-founded** when there is no infinite sequence $(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $\forall n, x_n \bowtie x_{n+1}$. Unless stated otherwise, the word **set** is used to designate a subset of $\mathbb{N}$, and **class** is used to designate a subset of $\mathfrak{P}(\mathbb{N})$.

Given $x, y \in \mathbb{N} \cup \{\pm\infty\}$, we define $[\![x, y]\!] := \{z \in \mathbb{N} : z \geqslant x \wedge z \leqslant y\}$. If a bracket symbol is flipped then its associated inequality is strict, e.g. $[\![x, y[\![:= \{z \in \mathbb{N} : z \geqslant x \wedge z < y\}$. Very often in the document, the integer $n \in \mathbb{N}$ is identified with the set $[\![0, n[\![$. This ambiguity should not cause any problem thanks to context. In particular, for any $k, \ell \in \mathbb{N}$, the set of maps $[\![0, k[\![^{[\![0,\ell[\![}$, is often identified with k^ℓ. Again, context should get rid of any ambiguity.

For any $n \in \mathbb{N}$, there is a bijection $\langle -, \dots, - \rangle$ from $\mathbb{N}^n$ to $\mathbb{N}$, which verifies $\forall x, y \in \mathbb{N}, \langle x_1, \dots, x_k \rangle \geqslant \max\{x_1, \dots, x_k\}$ and which is increasing on each variable. Given a set A, we denote by χ_A its **characteristic function**, i.e. $\forall x, x \in A \Leftrightarrow \chi_A(x) = 1$ and $x \notin A \Leftrightarrow \chi_A(x) = 0$. Often in the document, a set is identified with its characteristic function.

The symbols $\exists^\infty x$ and $\forall^\infty x$ are abbreviations for $\forall y, \exists x > y$ and $\exists y, \forall x > y$ respectively. The former should be read as "there are infinitely many x" and the latter as "for all but finitely many x". Note that they are the dual of each other.

Given two

1.1.2 Strings and sequences

Often in the document, a set is identified with the infinite binary sequence corresponding to its characteristic function. More genrally, we may identify functions from $\mathbb{N}$ to $\mathbb{N}$ with infinite sequences of integers, i.e. $f : \mathbb{N} \to \mathbb{N}$ corresponds to $\big(f(n)\big)_{n\in\mathbb{N}}$. Infinite sequences and sets are both denoted by capital Latin letters X, Y, Z, A, etc.

A **string** is a finite sequence of integers. The set of all strings is denoted $\mathbb{N}^{<\mathbb{N}} := \bigcup_{\ell\in\mathbb{N}} \mathbb{N}^\ell$. More particularly, given $k \in \mathbb{N}$, the strings composed only of integers within the set k are called **k-valued strings**. The set of all k-valued strings is denoted $k^{<\mathbb{N}} := \bigcup_{\ell\in\mathbb{N}} k^\ell$. For $k = 2$, such strings are called **binary strings**. We will generally denote strings by Greek letters σ, τ, μ, ρ, etc.

The **length** of a given string $\sigma : \ell \to \mathbb{N}$ is the integer $|\sigma| := \ell$. The set of strings, resp. k-valued strings, of length less than $n \in \mathbb{N}$ is denoted $\mathbb{N}^{<n} := \bigcup_{\ell<n} \mathbb{N}^\ell$, resp. $k^{<n} := \bigcup_{\ell<n} k^\ell$. The **empty string** $\varnothing \to \mathbb{N}$ is denoted ε. Given two strings σ and τ of respective length ℓ and m, we define their **concatenation** as the finite sequence $\sigma \cdot \tau : \ell + m \to \mathbb{N}$ which, given $j < \ell + m$, equals $\sigma(j)$ if $j < \ell$, and $\tau(j - \ell)$ otherwise. Concatenation is an associative operation, so a string $\sigma : \ell \to \mathbb{N}$ can be written as $\sigma(0) \cdot \ldots \cdot \sigma(\ell - 1)$. For any infinite sequence X and $\ell \in \mathbb{N}$, we define $X{\restriction_\ell} := X(0) \cdot \ldots \cdot X(\ell - 1)$

The **prefix relation** on strings, denoted $\prec$, is defined by $\sigma \prec \tau \iff |\sigma| < |\tau| \wedge \forall j < |\sigma|, \sigma(j) = \tau(j)$. It is a strict partial order, whose reflexive closure is denoted by $\preccurlyeq$. If two strings σ and τ are **incomparable** for $\prec$, we write $\sigma|\tau$. Moreover, for any string σ and infinite sequence X, we also write $\sigma \prec X$ to mean that σ is an initial segment of X, i.e. $X\restriction_{|\sigma|} = \sigma$. Finally, the **cylinder** of a string σ is $[\sigma] \coloneqq \{X \in 2^{\mathbb{N}} : \sigma \prec X\}$.

1.2 Computability theory

We begin this section with a short historical introduction on the subject. For more information see [Soa16, Part V]. The rest of the section gives formal definitions and fundamental results regarding basic notions in computability theory. For a more in-depth study, see [Soa16], [MP22] or [DM22].

1.2.1 History

At the beginning of the 20^{th} century, mathematicians became increasingly interested in the familiar notion of "procedure". They were motivated by the need for a more solid and formal foundation for mathematics, due to paradoxes in set theory that were uncovered by Burali-Forti and Russell. In particular, Hilbert had the hope of solving some questions in an algorithmic fashion. His famous tenth problem in 1900, and the *Entscheidungsproblem*[1] in 1928 are such examples.

Hence some mathematicians started to devise formal objects to try and capture the notion of algorithm. This led to the creation of μ-recursive functions, λ-calculus and Turing machines. These three models turned out to all be equivalent, i.e. whichever task one can do, any other can do as well. And since Turing had managed to convince everyone that his concept could be mechanized, it became accepted that each of these models captures exactly our informal notion of algorithm. This statement is known today as the **Church-Turing thesis**, and any model that is as powerful as one of those aforementioned is said to be **Turing complete**.

From there, computability theory did not need to be tied to any particular model

[1] German for "decision problem". Informally, the problem asks for an algorithm to decide whether or not a given mathematical statement is true or false.

of computation, in other words, paraphrasing a famous quote: "Computability theory is no more about computation models than astronomy is about telescopes." Hence, when we talk about "algorithms" or "computable functions", the reader should simply picture a list of instructions, written in their favorite programming language.

Figure 1.1: From left to right: Turing, Gödel, Church, Post

1.2.2 Basic notions

In computability theory, the main objects manipulated are functions from $\mathbb{N}$ to $\mathbb{N}$. Intuitively, an object is *computable* if its most basic features can be determined by an algorithm. For example, a function $f : \mathbb{N} \to \mathbb{N}$ is **computable** if there is an algorithm which outputs $f(n)$ when given n for input. A set of integers is computable if there is an algorithm capable of deciding, whether any integer belongs to the set or not.

Definition 1.2.1. A set $A \subseteq \mathbb{N}$ is **computable** if its characteristic function χ_A is computable.

Before continuing, we describe the general notation used for computable functions throughout this document. Firstly, we suppose that we have defined a bijection between $\mathbb{N}$ and all the computer programs that take an integer as input and possibly output an integer. Hence, we can manipulate programs with integers called **indexes** and typically denoted e, i or j. From there, $\Phi_e : \mathbb{N} \to \mathbb{N}$ is the (possibly partial) function defined by the program e. If the algorithm e stops in $t \in \mathbb{N}$ **steps of computation**[2] or less, on the input x, then we write $\Phi_e(x)[t]\downarrow$ and

[2] Informally, a computation step corresponds to an atomic unit of time. This idea can be made precise by the formalism of Turing machines, or by the workings of an actual computer.

$\Phi_e(x)[t]$ is the value of the computation, otherwise we write $\Phi_e(x)[t] \uparrow$. Moreover, if e eventually stops on the input x, i.e. $\exists t, \Phi_e(x)[t] \downarrow$, then we say that $\Phi_e(x)$ **converges** and we write $\Phi_e(x) \downarrow$. Otherwise we say that $\Phi_e(x)$ **diverges**, i.e. $\forall t, \Phi_e(x)[t] \uparrow$, and we write $\Phi_e(x) \uparrow$.

1.2.3 Computably enumerable sets and the halting problem

Quite quickly after defining computable sets, one can realize that there exist some sets that are not computable. Indeed, on one hand Cantor had previously shown, by using his famous diagonal argument, that there are uncountably many sets of integers, i.e. $\operatorname{card}(\mathfrak{P}(\mathbb{N})) > \operatorname{card}(\mathbb{N})$. On the other hand, there are only countably many algorithms, since a computer program consists of a finite sequence of characters. This strict difference in cardinality implies the existence of non-computable sets.

This argument is relatively simple, yet it does not provide any explicit example of a non-computable set. The logicians of the early 20^{th} century came up with different algorithmic problems they showed to be undecidable. Nowadays the most famous is the **halting problem** for Turing machines[3]: does a Turing machine eventually stop on a given input? This problem is undecidable, i.e. no algorithm can solve this question in its whole generality.

Definition 1.2.2. The **halting set** is $K \coloneqq \{e \in \mathbb{N} : \Phi_e(e) \downarrow\}$

The choice of definition for the halting set is justified by Cantor's diagonal argument, which is used in the proof that it is non-computable (see [Soa16, Theorem 1.6.5]).

Proposition 1.2.3. *The halting set is not computable.*

Even though no algorithm can solve the halting problem, it is natural to imagine one that partially solves it. Indeed, to see if an algorithm stops or not on some input, we can simply execute it and wait to see if it eventually stops. If it does,

[3] Contrary to what many people believe, this problem was not studied by Turing in his 1936 paper *On Computable Numbers With an Application to the Entscheidungsproblem* [Tur36]. Indeed, in this paper, Turing actually proves the undecidability of three problems: the "satisfaction" problem, the "printing" problem, and the Entscheidungsproblem. For more information see [Luc21].

then we can answer "yes", otherwise we will wait forever. Hence, with enough time, it is possible to list all the elements of the halting set: we execute more and more algorithms in parallel[4], and the ones that are in fact members of K will eventually reveal themselves. This property is encompassed in the following definition.

Definition 1.2.4. A set $A \subseteq \mathbb{N}$ is **computably enumerable**, written c.e., if there is an index e such that $\forall n, (n \in A \iff \Phi_e(n) \downarrow)$.
Similarly to computable sets, the program e that corresponds to a c.e. set is called **c.e. code**, and the set itself is denoted W_e.

Computable enumerability is a core notion of computability theory. Here is a fundamental result concerning c.e. sets.

Proposition 1.2.5 (Complementation theorem)**.** *If A and $\overline{A}$ are both c.e., then A is computable.*

For a proof see [Soa16, Theorem 2.1.14]. In particular, this proves that the complement $\overline{K}$ of the halting set is not c.e., since K is c.e. but not computable. We will later see that we can build sets of arbitrary complexity from the halting problem.

1.2.4 Oracles

What if a program could somehow compute a non-computable set? What else could be computed from there? This idea leads to the notion of **oracle**, i.e. a set $A \subseteq \mathbb{N}$ for which a program can ask, for any integer $n \in \mathbb{N}$, if $n \in A$ or not. Intuitively it can be seen as an infinite extra memory the program has access to. Of course having a computable set as an oracle does not give us any more computing power, since any access to it can be replaced by the execution of an algorithm. But having a non-computable set as an oracle does give more computing power. For example, having the halting set as an oracle gives the ability to know if a given algorithm stops or not on a given input, and so decisions can be made based on this new knowledge.

In terms of notation, the function whose code is e and that uses A as an oracle is called a **Turing functional** or simply **functional**, and is written Φ_e^A (or

[4] This is possible because Turing-complete models are not limited by memory.

sometimes $\Phi_e(A)$). The set with c.e. code e and oracle A is written W_e^A. More generally, definitions and proofs that do not use oracles can be **relativized**, i.e. modified so that they use oracles. For example, given an oracle A, "computable" becomes "A-computable", and c.e. becomes "A-c.e.".

For any functional Φ and any oracle A, we denote by Φ^A the function $n \mapsto \Phi^A(n)$. Moreover, if Φ^A is a total function from $\mathbb{N}$ to 2, then it is identified with the set $\{n \in \mathbb{N} : \Phi^A(n) \downarrow= 1\}$. The functional Φ is **total** if the function Φ^B is total for all set $B \subseteq \mathbb{N}$.

It is also possible to use binary strings as "incomplete oracles". The notation remains identical, i.e. we write Φ_e^σ to designate the functional of code e which uses $\sigma \in 2^{<\mathbb{N}}$ as an oracle. The main difference is that, on an input n, if the program asks the oracle about a value that is outside of its range, then the computation diverges, i.e. $\Phi_e^\sigma(n)\uparrow$. Note that if a computation stops on an oracle $X \subseteq \mathbb{N}$, then only a finite amount of it has been used. This fact is known as the **use principle**. In other words

$$\forall X, \forall e, n, \left(\Phi_e^X(n)\downarrow \iff \exists \sigma \prec X, \Phi_e^\sigma(n)\downarrow\right)$$

Formally, only one oracle is allowed, but this is in fact equivalent to having finitely many, thanks to the join operation.

Definition 1.2.6. The **join** of two sets A and B is the set

$$A \oplus B \coloneqq \{2n : n \in A\} \cup \{2n+1 : n \in B\}$$

Thus, instead of two oracle A and B, we can consider a single $A \oplus B$. Indeed, it contains all the information of both A and B, since $n \in A \iff 2n \in A \oplus B$ and $n \in B \iff 2n+1 \in A \oplus B$.

1.2.5 Turing reduction and Turing jump

With oracles, we can now define new operators and relations on sets. One of them corresponds to the relative complexity between two sets.

Definition 1.2.7. A set $A \subseteq \mathbb{N}$ is **Turing reducible** to another set $B \subseteq \mathbb{N}$, written $A \leqslant_T B$, if there is a program e such that $\Phi_e^B = A$.

Basically, if we knew how to compute B, then we could compute A. In particular, $A \leqslant_T \varnothing$ means that A is computable, by definition. The Turing reduction is a preorder, i.e. a reflexive and transitive relation.[5] Hence we can consider the equivalence relation generated by $\leqslant_T$, it is called the **Turing equivalence** relation, and is denoted by the symbol $\equiv_T$. The equivalent classes of this relation are called **Turing degrees**, and are denoted by bold lowercase letters such as $\mathbf{d}$, $\mathbf{a}$, $\mathbf{b}$, etc. If the equivalence class of a set X is $\mathbf{d}$, we may say "the set X of degree $\mathbf{d}$".

There is an operator on sets that generalizes the construction of the halting set.

Definition 1.2.8. The **Turing jump** (or simply **jump**) of a set A is the set

$$A' := \left\{ e \in \mathbb{N} : \Phi_e^A(e) \downarrow \right\}$$

Remark 1.2.9. Note that for $A = \varnothing$, we obtain $\varnothing' = K$. This new notation of the halting set is what we use from now on.

A basic property of the jump is that it is compatible with the Turing reduction, i.e. for any sets X and Y, if $X \leqslant_T Y$ then $X' \leqslant_T Y'$.

Another basic property comes from the proof of Proposition 1.2.3. It can be relativized to prove that the jump of a set is strictly more complicated than the set itself.

Proposition 1.2.10. *For any $A \subseteq \mathbb{N}$, we have $A' >_T A$, i.e. $A' \geqslant_T A$ and $A' \nleqslant_T A$.*

Hence by taking the jump repeatedly, we get sets that are increasingly more complicated. For this purpose, we define iterated Turing jumps.

Definition 1.2.11. The **iterated Turing jumps** of a set $A \subseteq \mathbb{N}$ are defined recursively by

- $A^{(0)} := A$
- $A^{(n+1)} := \left(A^{(n)}\right)'$

We will soon see that the iterated jumps of $\varnothing$ form the backbone of the complexity hierarchy of sets.

[5] However it is not a partial order, because it is not antisymmetric. Indeed, the set of even numbers and the set of odd numbers are both computable, so they can compute each other, yet they are different.

Low sets

Regarding the Turing jump, we know that if $X \subseteq \mathbb{N}$ is computable, i.e. $X \leqslant_T \varnothing$, then $X' \equiv_T \varnothing'$. It is natural to ask whether or not computable sets are the only sets whose jump is equivalent to $\varnothing'$. This question admits a positive answer and leads to a fundamental notion in computability theory.

Definition 1.2.12. A set $X \subseteq \mathbb{N}$ is **low** if $X' \leqslant_T \varnothing'$.

Note that, for any set X we have $\varnothing \leqslant_T X$, which implies $\varnothing' \leqslant_T X'$. Proving the existence of a low set can be done by using a forcing argument, see [MP22, Proposition I.9.1].

1.2.6 The arithmetic hierarchy and Post's theorem

There is another way of assessing the complexity of a set $A \subseteq \mathbb{N}$, based on how many quantifiers alternations are needed to define it. It is called the **arithmetic hierarchy**[6].

Definition 1.2.13 (Arithmetic hierarchy)**.** Let X be an oracle and let $n \geqslant 1$. We define $\Sigma^0_n(X)$ to be the class of sets of the form

$$\big\{y \in \mathbb{N} : \exists x_1, \forall x_2, \ldots, Q x_n, (y, x_1, \ldots, x_n) \in R\big\}$$

where $R \subseteq \mathbb{N}^{n+1}$ is X-computable, and Q is the symbol $\exists$ if n is odd, and $\forall$ if n is even.
Accordingly, $\Pi^0_n(X)$ is the class of sets of the form

$$\big\{y \in \mathbb{N} : \forall x_1, \exists x_2, \ldots, Q x_n, (y, x_1, \ldots, x_n) \in R\big\}$$

where $R \subseteq \mathbb{N}^{n+1}$ is X-computable, and Q is the symbol $\exists$ if n is even, and $\forall$ if n is odd.
We also define the class $\Delta^0_n(X) \coloneqq \Sigma^0_n(X) \cap \Pi^0_n(X)$. If $X = \varnothing$, then we simply write Σ^0_n, Π^0_n and Δ^0_n.

The halting set $\varnothing'$ is an example of Σ^0_1 set, because it can be written as $\{e \in \mathbb{N} : \exists t, \Phi_e(e)[t]\downarrow\}$. What really counts is the alternation of quantifiers. For instance, if

[6] The name comes from the fact that the sets of this hierarchy correspond exactly to the sets that are definable by a first-order formula in the language of Peano arithmetic.

a set can be described by a formula of the form $\exists x_1, \exists x_2, R(y, x_1, x_2)$, then x_1 and x_2 can be encoded by a pair, and so there is another predicate $S \subseteq \mathbb{N}^2$ such that the formula $\exists x_3, S(y, x_3)$ describes the set as well. The predicate S is basically the same as R but considers that x_3 represents the pair $\langle x_1, x_2 \rangle$.

Right away we can see that if a formula describes a $\Sigma_n^0(X)$ set, then its negation corresponds to a $\Pi_n^0(X)$ set.

Proposition 1.2.14. *Let X be an oracle. For any $A \subseteq \mathbb{N}$ we have*

$$A \in \Sigma_n^0(X) \iff \overline{A} \in \Pi_n^0(X)$$

Choosing the halting set as an example earlier was not arbitrary. Indeed, there is an important correspondence between the classes of the arithmetic hierarchy and the iterations of the Turing jump on $\varnothing$.

Theorem 1.2.15 (Post)**.** *Let $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$*

$$A \text{ is } \varnothing^{(n)}\text{-computable} \iff A \text{ is } \Delta_{n+1}^0$$
$$A \text{ is } \varnothing^{(n)}\text{-c.e.} \iff A \text{ is } \Sigma_{n+1}^0$$

For a proof of the theorem, see [Soa16, Theorem 4.2.2]. In particular the class Δ_1^0 corresponds exactly to computable sets, the class Σ_1^0 corresponds exactly to c.e. sets, and the class Δ_2^0 corresponds exactly to sets that can be computed from $\varnothing'$.

What is more, this theorem proves that the arithmetic hierarchy is strict, i.e. for any $n \in \mathbb{N}$, there is a set in any class $\Gamma \in \{\Sigma_n^0, \Pi_n^0, \Delta_n^0\}$ which is not in any of the two other classes. Indeed, by using Proposition 1.2.5 we can prove that the set $\varnothing^{(n)}$ is Σ_n^0 but not Π_n^0, and the set $\overline{\varnothing^{(n)}}$ is Π_n^0 but not Σ_n^0. Then the set $\varnothing^{(n)} \oplus \overline{\varnothing^{(n)}}$ is a strict Δ_{n+1}^0 set, as otherwise we could give a Σ_n^0 formula that characterizes the elemebts of $\overline{\varnothing^{(n)}}$, or a Π_n^0 formula that characterizes the elements of $\varnothing^{(n)}$.

A nice visual representation of the arithmetic hierarchy emerges from all these properties. See Section 1.2.6.

1.2.7 Approximations and Shoenfield's lemma

It is quite natural to approximate a given set using a sequence of sets. We are then interested in establishing how complex such a set can be, depending on the

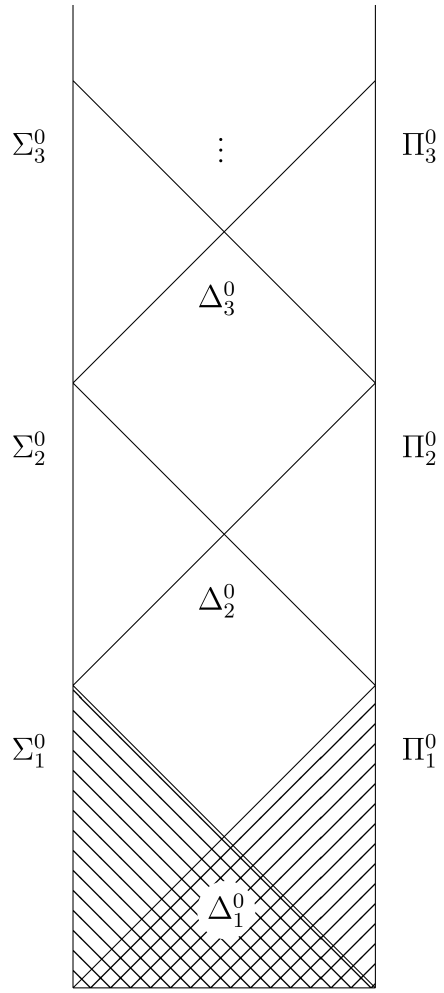

Figure 1.2: A representation of the arithmetic hierarchy

complexity of the sequence. For the next definition recall that a set is identified with its characteristic function.

Definition 1.2.16. A sequence of sets $(A_n)_{n\in\mathbb{N}}$ is an **approximation** of a set $A \subseteq \mathbb{N}$, if $\forall x, A(x) = \lim_n A_n(x)$.

A sequence of sets $(A_n)_{n\in\mathbb{N}}$ is represented by a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\forall n, f(n, x) = A_n(x)$. In particular, if f is computable then $(A_n)_{n\in\mathbb{N}}$ is called a **computable sequence**.

Some classes of sets can be characterized in terms of approximations. This is the case for computably enumerable sets. Indeed, if a set $A \subseteq \mathbb{N}$ is c.e. with code

e, then the sequence $(A[t])_{t\in\mathbb{N}}$ is an approximation of A, where $A[t] \coloneqq \{x \in \mathbb{N} : \Phi_e(x)[t]\downarrow\}$. This example leads to the following definition.

Definition 1.2.17. A computable sequence of sets $(A_n)_{n\in\mathbb{N}}$ is a **c.e. approximation** of a set $A \subseteq \mathbb{N}$, if $\forall n, A_n \subseteq A_{n+1}$ and $A = \bigcup_n A_n$.

Having a c.e. approximation is equivalent to having an approximation that starts with the empty set, and progressively adds elements, the restriction being that an element can never be removed.

Proposition 1.2.18. *A set $A \subseteq \mathbb{N}$ is c.e. if and only if it has a c.e. approximation.*

The class of Δ^0_2 sets also possesses a characterization that is both useful and natural.

Definition 1.2.19. A set $A \subseteq \mathbb{N}$ is **limit computable** if it has a computable approximation.

Theorem 1.2.20 (Limit lemma, Shoenfield)**.** *The class of limit computable sets is exactly the class of $\varnothing'$-computable sets.*

1.2.8 Hyperimmunity

We now approach an important family of functions, exploiting the idea that the computational complexity of a function is closely linked to its rate of growth.

Definition 1.2.21. A function $f : \mathbb{N} \to \mathbb{N}$ **dominates** a function $g : \mathbb{N} \to \mathbb{N}$ if $\forall^\infty x, f(x) \geqslant g(x)$. A function $g : \mathbb{N} \to \mathbb{N}$ is **hyperimmune** if it is not dominated by any computable function.

Note that if f is a computable function then so is $x \mapsto f(x) + 1$, thus a function g is hyperimmune if and only if, for any total computable function, we have $\exists^\infty x, g(x) > f(x)$. So a hyperimmune function can grow relatively slowly, as long as it has infinitely many “spikes” of high value.

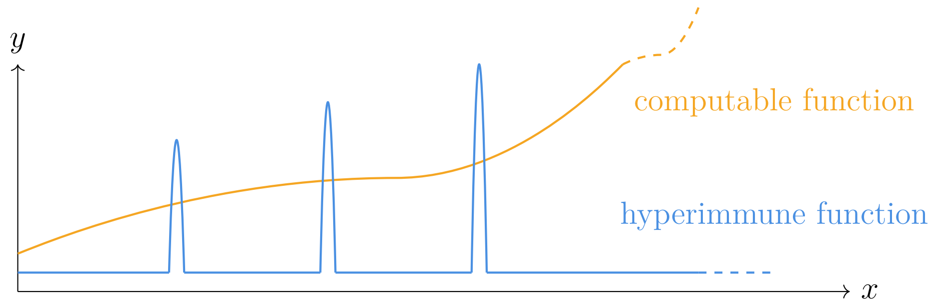

Figure 1.3: Representation of a hyperimmune function

There is another notion of hyperimmunity, for infinite sets this time. The idea is that a set A could be complex enough so that we would be unable to computably list finite sets of integers in which there is always at least one element of A.

Definition 1.2.22. We fix a canonical listing of the finite sets $(D_n)_{n\in\mathbb{N}}$. A **c.e. array** is a sequence of mutually disjoint finite sets $(D_{f(n)})_{n\in\mathbb{N}}$ where $f:\mathbb{N}\to\mathbb{N}$ is a computable function. An infinite set $A\subseteq\mathbb{N}$ is **hyperimmune** if for every c.e. array $(D_{f(n)})_{n\in\mathbb{N}}$, there is some $m\in\mathbb{N}$ such that $D_{f(m)}\cap A=\varnothing$. A Turing degree is **hyperimmune** if it contains a hyperimmune set, otherwise it is **hyperimmune-free**.

There is a link between the two definitions of hyperimmunity that we have seen. Basically, a set is hyperimmune if the function that corresponds to its sparcity is hyperimmune.

Definition 1.2.23. The **principal function** of an infinite set $A := \{a_0 < a_1 < \dots\}$ is the function $p_A:\mathbb{N}\to\mathbb{N}$ such that $\forall n, p_A(n)=a_n$.

Proposition 1.2.24 (Kuznetsov, Medvedev, Uspensky [Usp63])**.** *An infinite set is hyperimmune if and only if its principal function is hyperimmune.*

For a proof see [Soa16, Theorem 5.3.3]. This equivalence also implies that a degree is hyperimmune-free if and only if it does not compute any function that is hyperimmune. This is the reason why the term "**computably dominated**" can also be used instead of "hyperimmune-free".

Finally, this last result provides a simple condition to obtain hyperimmune sets.

Theorem 1.2.25 ([MM68, Theorem 1.2]). *If a set $A \subseteq \mathbb{N}$ is Δ^0_2 and non-computable, i.e. $\varnothing <_T A \leqslant_T \varnothing'$, then it is hyperimmune.*

1.2.9 Trees and Π^0_1-classes

In this section, we focus our study on classes of sets, their complexity, and how they relate to trees. We will briefly refer to topology as we discuss some aspects of Cantor space, but no prior knowledge of this field is required here.

Cantor space

The class $2^{\mathbb{N}}$ of infinite binary sequences is called **Cantor space**[7]. There is a correspondence between Cantor space and the unit real interval $[0, 1]$, in the sense that any $X \in 2^{\mathbb{N}}$ can be associated to the real whose binary representation is $0.X_0X_1X_2\ldots$, and any real of $[0, 1]$ has a binary representation that can be seen as an infinite binary sequence[8]. Cantor space can be equipped with a topology based on the cylinders of binary strings, i.e. the classes of the form $[\sigma] := \{X \in 2^{\mathbb{N}} : X \succ \sigma\}$ where σ is a binary string. See Figure 1.4.

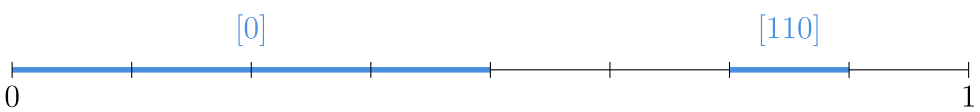

Figure 1.4: A representation of the cylinders [0] and [110] on the unit interval. Informally, "0" can be thought of as "take the left half of the interval you are in" and "1" as "take the right half of the interval you are in".

[7] This is because there is a one-to-one correspondence between $2^{\mathbb{N}}$ and the famous "Cantor ternary set".

[8] Note that some reals actually have two binary representations. For example $0.1000\ldots$ and $0.0111\ldots$ correspond to the same real. This situation is analog to the well-known equality $0.999\ldots = 1$.

Borel hierarchy and Lebesgue measure

An **open class** is an arbitrary union of cylinders, i.e. a class of the form $[W] := \bigcup_{\sigma \in W} [\sigma]$ where $W \subseteq 2^{<\mathbb{N}}$, and a **closed class** is the complement of an open class. The collection of open classes and closed classes are respectively denoted $\utilde{\Sigma}^0_1$ and $\utilde{\Pi}^0_1$. They form the first level of the **Borel hierarchy**. A class is then $\utilde{\Sigma}^0_{n+1}$ if it is a countable union of $\utilde{\Pi}^0_n$ classes, and it is $\utilde{\Pi}^0_{n+1}$ if it is a countable intersection of $\utilde{\Sigma}^0_n$ classes. A class in this hierarchy is called a **Borel class**.

The Lebesgue measure on the unit interval, denoted μ in this document, can be seen as a measure on Borel classes. It is defined as the unique measure such that $\mu\big([\sigma]\big) = 2^{-|\sigma|}$ for any cylinder $[\sigma]$. Carathéodory's extension theorem ensures the existence and uniqueness of this measure.

Effective classes

In computability theory, we are interested in an effective equivalent of these notions, where effective means that at least some aspects of these objects can be manipulated computationally. For example, cylinders are convenient to manipulate, because they rely on a string, which is a finite object. Open and closed classes can also be manipulated effectively.

Definition 1.2.26. A Σ^0_1**-class** is a class of the form $[W_e] := \bigcup_{\sigma \in W_e} [\sigma]$ where W_e is a c.e. set of binary strings. And a Π^0_1**-class** is the complement of a Σ^0_1-class. The **code** of a Σ^0_1-class (or Π^0_1-class) is the c.e. code of its underlying set W_e.

The following characterization is useful to prove that some classes are Σ^0_1 or Π^0_1.

Proposition 1.2.27. *Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be a class.*

- *$\mathcal{C}$ is Σ^0_1 if and only if it is of the form $\{X \in 2^{\mathbb{N}} : \exists n, X{\upharpoonright}_n \in R\}$, where $R \subseteq 2^{<\mathbb{N}}$ is a computable set of strings.*
- *$\mathcal{C}$ is Π^0_1 if and only if it is of the form $\{X \in 2^{\mathbb{N}} : \forall n, X{\upharpoonright}_n \in R\}$, where $R \subseteq 2^{<\mathbb{N}}$ is a computable set of strings.*

Trees

We now introduce the classic structure of finitely branching trees, which are closely related to Π^0_1-classes.

Definition 1.2.28. A **tree** is a set of strings $T \subseteq k^{<\mathbb{N}}$ (for some $k \in \mathbb{N}$) which is downward-closed for $\prec$, i.e. $\forall \sigma \in T, \tau \prec \sigma \implies \tau \in T$. For $k = 2$, such trees are called **binary trees**.

The elements of a tree are generally called **nodes**. When two nodes $\alpha, \beta \in T$ are such that $\alpha \prec \beta$ we say that β is a **successor** of α, or that α is an **ancestor** (or **predecessor**) of β. A node β is a **direct successor** (or **child**) of α if there is $n \in \mathbb{N}$ such that $\beta = \alpha \cdot n$. A node with no successor is called a **leaf** (or **terminal node**). The set of all the leaves of a tree T is denoted $\ell(T)$. A representation of a tree is given in Figure 1.5.

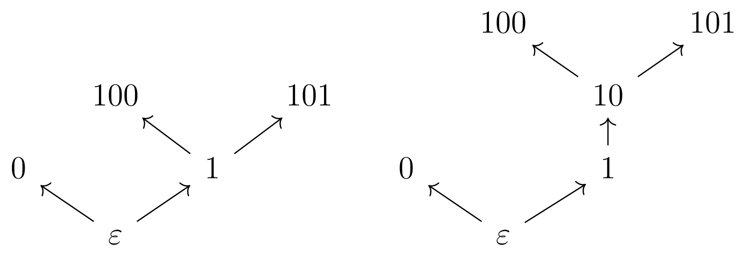

Figure 1.5: The left structure does not represent a tree, whereas the right one does.

Definition 1.2.29. An **infinite path** (or simply **path**, or **branch**) of a tree $T \subseteq k^{<\mathbb{N}}$ is an infinite sequence $X \in k^{\mathbb{N}}$ such that $\forall \ell \in \mathbb{N}, X{\upharpoonright}_\ell \in T$. The class of infinite paths of a tree T is denoted $[T]$. Similarly, a **finite path** is a string $\sigma \in k^{<\mathbb{N}}$ such that $\forall \ell \in \mathbb{N}, \sigma{\upharpoonright}_\ell \in T$. A tree with no infinite path is **well-founded**.

A tree $T \subseteq k^{<\mathbb{N}}$ is **computable** if T is computable as a set, i.e. if there is an algorithm that can decide, for any $\sigma \in k^{\mathbb{N}}$, if $\sigma \in T$ or not.[9] Hence, the **code** of a computable tree T refers to $e \in \mathbb{N}$ such that $\Phi_e = T$. It turns out there is a link between the infinite paths of computable trees and Π^0_1-classes.

[9] To be more rigorous we should consider a bijection between strings and integers, then T can be seen as a set of integers thanks to that bijection.

Proposition 1.2.30. *A class $\mathcal{C}$ is Π^0_1 if and only if there is a computable binary tree T such that $\mathcal{C} = [T]$*

What is more, the proof of Proposition 1.2.30 (see [DM22, Proposition 2.8.7]) is in fact **uniform**, i.e. there is a computable procedure which, given the code of a Π^0_1-class as an input, outputs the code of a computable tree associated to it. There is also another computable procedure that does the converse. This means that trees and Π^0_1-classes can be used interchangeably.

1.2.10 Basis theorems

Since infinite trees are structures that often arise in computability theory, it is useful to know how complex their paths can be. Basis theorems provide some insight regarding this question and thus are at the heart of many techniques.

Definition 1.2.31. A class $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is a **basis** for Π^0_1-classes if any non-empty Π^0_1-class contains an element of $\mathcal{C}$.

In essence, basis theorems point out classes that are basis, meaning that, for any computable tree T such that $[T]$ is not empty, we can find an infinite path that verifies a certain property. We now list the most famous basis theorems.

Theorem 1.2.32 (Kreisel)**.** *The class of Δ^0_2 sets is a basis for Π^0_1-classes.*

Theorem 1.2.33 (Low basis [JS72a, Theorem 2.1])**.** *The class of low sets is a basis for Π^0_1-classes.*

Theorem 1.2.34 (Computably dominated (or hyperimmune-free) basis [JS72a, Theorem 2.4])**.** *The class of computably dominated sets is a basis for Π^0_1-classes.*

Definition 1.2.35. The **cone** above a set $X \in \mathbb{N}$ is the class

$$\{Y \subseteq \mathbb{N} : Y \geqslant_T X\}$$

Theorem 1.2.36 (Cone avoidance basis [JS72a, Theorem 2.5])**.** *Given a non-computable set X. The complement of the cone above X is a basis for Π^0_1-classes.*

1.2.11 PA degrees

We now discuss an important family of Turing degrees, namely PA degrees, where PA stands for Peano Arithmetic, the famous set of axioms of first-order logic that was at the center of many preoccupations for the logicians at the beginning of the 20^{th} century. Before going any further, we must introduce some concepts relative to proof theory, without entering deeply into the details.

A **theory** is simply a set of axioms. If a formula φ is **provable**[10] from a set of axioms T, we write $T \vdash \varphi$, if it is not we write $T \nvdash \varphi$. More generally, given a set of formulas F, we write $T \vdash F$ instead of $\forall \varphi \in F, T \vdash \varphi$, and $T \nvdash F$ instead of $\exists \varphi \in F, T \nvdash \varphi$. A theory is **consistent** if it does not prove a contradiction, i.e. it does not prove both a formula and its negation.[11] It is called **complete** if, for any formula φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$, i.e. anything can be either proved or disproved. A theory T' is a **completion** of another theory T if $T' \supseteq T$ and T' is complete.

A formula of arithmetic can be encoded by an integer, just like computer programs. It is even possible to computably enumerate all the formulas of arithmetic. We fix such an enumeration $(\varphi_n)_{n \in \mathbb{N}}$, and call n the **code** of the formula. A theory can then be represented as a set of integers. We are now able to give the historical definition of PA degrees.

Definition 1.2.37. A Turing degree is PA if it contains a consistent completion of Peano arithmetic.

[10] A set of axioms T proves a formula φ if there is a sequence of formulas such that, the last formula is φ, and each formula is either an axiom or can be obtained from the previous formulas by applying a rule of inference from a given logic.

[11] Equivalently, since any formula can be proven from a contradiction (a principle known as "ex falso quodlibet"), the theory must not prove all formulas.

The notion of PA degree is very robust, in the sense that it has many different characterizations, see [Soa16, Theorem 10.3.3]. One of them in particular relates PA degrees to non-empty Π^0_1-classes.

Theorem 1.2.38 (Scott [Sco62])**.** *A set $X \subseteq \mathbb{N}$ is of PA degree if and only if it computes a set in any non-empty Π^0_1-class.*

There is an alternative definition for PA degrees resulting from this theorem, it is the one that is generally used nowadays, notably because it can be relativized.

Definition 1.2.39. Let A be an oracle. A Turing degree is $\mathsf{PA}(A)$ if the sets that are A-computable from it form a basis for Π^0_1-classes. If a set $X \subseteq \mathbb{N}$ is of $\mathsf{PA}(A)$ degree, we write $X \ll A$.

Finally, we present another useful characterization of PA degrees, relating them to some other remarkable Turing degrees.

Definition 1.2.40. A function $f : \mathbb{N} \to 2$ is **diagonally non-computable** if $\forall n, f(n) \neq \Phi_n(n)$. A Turing degree is DNC_2 if it computes a diagonally non-computable function.

Theorem 1.2.41 (Jockusch and Soare [Joc72a])**.** *A set $X \subseteq \mathbb{N}$ is of PA degree if and only if it is of DNC_2 degree.*

In particular, this theorem leads to another useful property of PA degrees regarding Π^0_1-classes. Indeed, the class of DNC_2 functions is a Π^0_1-class, as it can be written as

$$\{f \in 2^{\mathbb{N}} : \forall e, \forall t, \Phi_e(e)[t] \neq f(e)\}$$

where the symbol $\neq$ is used as a shorthand for $\Phi_e(e)[t] \uparrow \vee \Phi_e(e)[t] \downarrow \neq f(e)$.

Proposition 1.2.42. *There exists a non-empty Π^0_1-class whose members are all of degree PA.*

This fact can be used together with any basis theorem, for example to deduce that there exists a low PA degree.

1.2.12 Forcing

Forcing is a powerful tool originating from set theory. It has been used with success in computability theory and reverse mathematics, where it has become a central tool. The apparatus necessary for forcing is much simpler in computability theory than in set theory, hence some definitions might differ from one field to the other.

The main technical feat of forcing is the possibility to construct sets while ensuring, *during the construction*, that they verify some properties. These properties are often called **requirements**. Informally, to construct a set G via forcing, we use successive approximations of that set, each extending the previous one. For example, if G is seen as an infinite binary sequence, then it can be obtained by constructing a sequence $(p_i)_{i \in \mathbb{N}}$ of binary strings such that $p_0 \preccurlyeq p_1 \preccurlyeq \dots$ A given approximation p can result in many different sets depending on how it is extended. We denote by $[p]$ the class of sets that are approximated by p. More formally:

Definition 1.2.43. A **notion of forcing** is a partial order $(\mathbb{P}, \leqslant)$, whose elements are called **conditions**, together with a non-decreasing function $[\cdot] : \mathbb{P} \to \mathfrak{P}(2^{\mathbb{N}})$ called the **interpretation**. For any $p, q \in \mathbb{P}$ such that $p \leqslant q$, we say that p **extends** q.
A **filter** for $\mathbb{P}$ is a non-empty class $\mathcal{F} \subseteq \mathbb{P}$ which is **upward-closed**, i.e. $\forall p \in \mathcal{F}, \forall q \in \mathbb{P}, (p \leqslant q \implies q \in \mathcal{F})$ and **compatible** i.e. $\forall p, q \in \mathcal{F}, \exists r \in \mathcal{F}, r \leqslant p \wedge r \leqslant q$.

Remark 1.2.44.

- We say that q extends p even though we write $q \leqslant p$, because $[q]$ is seen as a set of potential candidates for the set we are constructing, and $[q] \subseteq [p]$ by definition, which means that we have narrowed down the number of potential candidates. Note that some authors use the opposite convention and write $q \geqslant p$.
- In this document, we will only need to construct infinite decreasing sequences of conditions instead of filters, and the two terms are used interchangeably. Moreover, such a sequence $(p_n)_{n \in \mathbb{N}}$ induces a filter $\{q \in \mathbb{P} : \exists n, p_n \leqslant q\}$.

The example that was mentioned above is called **Cohen forcing**, it is the first notion of forcing that was ever developed. In this case, $\mathbb{P}$ is the set of binary strings, $\rho \leqslant \sigma$ if and only if $\rho \preccurlyeq \sigma$ the prefix relation on them, and the interpretation of a string is its cylinder.

Given a filter $p_0 \geqslant p_1 \geqslant \dots$, the definition of the interpretation yields that $[p_0] \supseteq [p_1] \supseteq \dots$ So the set G that we want to construct is taken from the class $\bigcap_{n\in\mathbb{N}}[p_n]$. For simpler notation, we will soon see how to make this intersection a singleton. The big question is "how to ensure that the filter we are constructing yields the desired properties on G?". As a first approach, consider Cohen forcing and the property "G contains an even number". If a condition p corresponds to a finite set that does contain an even number, then any extension of p will also contain an even number, thus p is "forcing" G to contain an even number. Therefore, given a property φ, we could try to build a filter that contains a condition p such that $\forall G \in [p], \varphi(G)$. However, this definition is too restrictive and fails to capture enough properties. Indeed, suppose we constructed a filter $\mathcal{F} \coloneqq p_0 \geqslant p_1 \geqslant \dots$ of Cohen forcing such that $\forall n, \operatorname{card}\left(\left\{i < |p_n| : p_n(i) = 1\right\}\right) = n$, and consider the property "G is infinite". A set resulting from $\mathcal{F}$ does verify this assertion, but our naive definition fails to realize this, because, for any $p \in \mathcal{F}$, its cylinder $[p]$ contains both finite and infinite sets. To unveil the right notion, we need some extra work.

Notice how, for the property "G contains an even number", it is possible at any point during the construction of our filter, to satisfy the formula with the next extension.

Definition 1.2.45. A class $\mathcal{D} \subseteq \mathbb{P}$ is **dense** in $\mathbb{P}$ if any condition can be extended by an element of $\mathcal{D}$, i.e. $\forall p \in \mathbb{P}, \exists q \in \mathcal{D}, q \leqslant p$. A filter $\mathcal{F}$ **meets** a dense class $\mathcal{D}$ if $\mathcal{F} \cap \mathcal{D} \neq \varnothing$.

Hence the class $\{\sigma \in 2^{<\mathbb{N}} : \exists n, \sigma(2n) = 1\}$ is dense for Cohen forcing. In the example stated above, the justification of "G is infinite" can now be reformulated: the filter $\mathcal{F}$ previously constructed meets the family of dense classes $(\mathcal{D}_n^{\text{inf}})_{n\in\mathbb{N}}$, where

$$\mathcal{D}_n^{\text{inf}} \coloneqq \left\{\sigma \in 2^{<\mathbb{N}} : \operatorname{card}\left(\left\{i < |\sigma| : \sigma(i) = 1\right\}\right) = n\right\}$$

This formulation leads us to the next definition.

Definition 1.2.46. Given a countable family of classes $\vec{\mathcal{D}} \coloneqq \{\mathcal{D}_i\}_{i\in\mathbb{N}}$, a filter $\mathcal{F}$ is **$\vec{\mathcal{D}}$-generic** if it meets every class of the family, i.e. $\forall i \in \mathbb{N}, \mathcal{F} \cap \mathcal{D}_i \neq \varnothing$.

In particular, for any notion of forcing $\mathbb{P}$, if a decreasing sequence of condition $\mathcal{F} \coloneqq (p_n)_{n\in\mathbb{N}}$ is is generic for the family $\vec{\mathcal{D}} \coloneqq \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ where $\mathcal{D}_n \coloneqq \{p \in \mathbb{P} : \exists \sigma \in 2^n, \forall X \in [p], X \succ \sigma\}$, then $\bigcap_{p\in\mathcal{F}}[p]$ is a singleton. Indeed, for any n there is $p_n \in \mathcal{F}$ and $\sigma_n \in 2^n$ such that $\forall X \in [p_n], X \succ \sigma_n$. Thus $\bigcap_{n\in\mathbb{N}}[p_n]$ is a singleton whose element is entirely determined by the sequence $(\sigma_n)_{n\in\mathbb{N}}$, and since

$\bigcap_{p\in\mathcal{F}}[p] \subseteq \bigcap_{n\in\mathbb{N}}[p_n]$ we have the desired result. For simplicity, we will always consider filters $\mathcal{F}$ that are generic for $\vec{\mathcal{D}}$, and denote the only element of $\bigcap_{p\in\mathcal{F}}[p]$ by $G_\mathcal{F}$. We can now define the forcing relation properly.

Definition 1.2.47. A condition $p \in \mathbb{P}$ **forces** a property φ, i.e. an arithmetical formula with a free second-order variable G, denoted $p \Vdash \varphi(G)$, if there is a countable family of dense classes $\vec{\mathcal{D}}$ such that, for all $\vec{\mathcal{D}}$-generic filter $\mathcal{F}$, if $p \in \mathcal{F}$ then $\varphi(G_\mathcal{F})$ holds.

Remark 1.2.48. We may sometimes talk about **sufficiently generic** filters to signify that such a filter will be determined a posteriori, once the family of dense classes for which it must be generic are fully known. The existence of such a filter will be guaranteed by Theorem 1.2.49.

The definition given here is referred to as the **semantic definition** of forcing, it is relatively simple to define and work with, but it quantifies over high-order objects such as families of dense classes and filters. Fortunately, there is a **syntactic definition** of forcing as well, much simpler in terms of complexity as it is based on induction, and which corresponds exactly to the semantic one. The details of this definition can be found in [DHR20, Definition 7.4.1].

The soundness of Definition 1.2.47 is assessed by the following facts. Firstly, as expected, if $p \Vdash \varphi(G)$ and $q \leqslant p$, then $q \Vdash \varphi(G)$. Secondly, the class $\{p \in \mathbb{P} : (p \Vdash \varphi(G)) \vee (p \Vdash \neg\varphi(G))\}$ is dense, meaning that we can always find an extension that decides a given property. In particular, for a sufficiently generic filter $\mathcal{F}$, $\varphi(G_\mathcal{F})$ holds if and only if $\exists p, p \Vdash \varphi(G)$. Thirdly, the following theorem ensures that, for any countable family of dense classes, we can carry out the construction of our generic filter. The idea of the proof is that, since there are countably many classes, and they are all dense, we can build a filter that meets them one by one.

Theorem 1.2.49 (Rasiowa and Sikorski)**.** *For any notion of forcing $\mathbb{P}$, any condition p, and any countable family of dense class $\vec{\mathcal{D}}$, there exists a $\vec{\mathcal{D}}$-generic filter containing p.*

Finally, we define a notion of forcing, called **Mathias forcing**, that is particularly well-suited for studying Ramsey's theorem. Many ulterior notions are refinements of Mathias forcing, where extra properties or objects are added to the condition.

Definition 1.2.50 ([Mat77])**.** A **Mathias condition** is a pair (σ, X) where
- σ is a binary string (identified with a finite set)
- $X \in 2^{\mathbb{N}}$ is an infinite set, called the **reservoir**
- $X \cap [\![0, |\sigma|[\![= \varnothing$

A Mathias condition (τ, Y) extends another (σ, X), if $Y \subseteq X$ and $\tau \succcurlyeq \sigma$ where $\tau - \sigma \subset X$. The interpretation of a Mathias condition is given by $\big[(\sigma, X)\big] \coloneqq \{Z \in [\sigma] : Z \subseteq \sigma \cup X\}$.

1.3 Reverse mathematics

1.3.1 Overview

Reverse mathematics is a foundational program started in 1974 by Harvey Friedman [Fri74]. The original goal was to answer the following question: "What are the weakest axioms required to prove a given theorem?". More generally, reverse mathematics provide a framework in which the tools of computability theory and proof theory are used to assess the constructive content of theorems, and how they interact with one another. Chasing these objectives also often leads to finding new proofs of known theorems, usually simpler in terms of axiomatic complexity. For more information on the subject see [Sim09], [MP22] or [DM22].

1.3.2 Second-order arithmetic

The formal framework used by reverse mathematics is second-order arithmetic. This choice is motivated by the fact that arithmetic and computability theory both deal with natural numbers at their core, and that a relevant chunk of mathematics can be expressed in second-order arithmetic. We shall call this chunk "ordinary" mathematics, as opposed to "set-theoretic" mathematics. For example, continuous function from $\mathbb{R}$ to $\mathbb{R}$ can be encoded by sets of integers, because continuous functions are entirely determined by their action on open sets, and $\mathbb{R}$ possesses a countable basis of open sets. More examples can be found in *Grundlagen der Mathematik* [HB11] or in Simpson's book *Subsystems of second order arithmetic* [Sim09]. We now dive into more formal definitions.

The language of second-order arithmetic is composed of first-order variables that represent natural numbers, and are denoted by lowercase letters like x, y, z. It also contains second-order variables that represent sets of natural numbers, and are denoted by uppercase letters like X, Y, Z. There are also all the necessary logical and arithmetical symbols: parenthesis, $\wedge, \vee, \implies, \neg, \forall, \exists, +, \times, =, <, 0, 1$. Finally, there is the set-theoretic symbol $\in$. Formulas are formed from this language in the same way as in first-order logic, the only difference is that quantification can occur on second-order variables as well. The theory of second-order arithmetic, written Z_2, is composed of **Robinson's arithmetic axioms**, denoted by Q.

$$\begin{array}{ll} \forall x, x+1 \neq 0 & \forall x, (x \neq 0 \implies \exists y, x = y+1) \\ \forall x, \forall y, (x+1 = y+1 \implies x = y) & \forall x, x+0 = x \\ \forall x, \forall y, x+(y+1) = (x+y)+1 & \forall x, x \times 0 = 0 \\ \forall x, \forall y, x \times (y+1) = (x \times y)+x & \forall x, \forall y, \big(x < y \iff \exists z, (z \neq 0 \wedge y = x+z)\big) \end{array}$$

Plus the comprehension scheme,

$$\exists X, \forall y, (y \in X \iff \varphi(y))$$

for any formula φ. And finally, the induction scheme

$$\varphi(0) \wedge \Big(\big(\forall x, (\varphi(x) \implies \varphi(x+1))\big) \implies \forall x, \varphi(x)\Big)$$

1.3.3 Models

Henkin models

A **Henkin model** of second-order arithmetic is a mathematical structure of the form $\langle N, S\rangle$, where $S \subseteq \mathfrak{P}(N)$, along with an **interpretation** of the symbols in the language of second-order arithmetic. The set N and the class S respectively correspond to the integers and the sets of integers in the model. So they are respectively called **first-order part** and **second-order part** of the model. The interpretation consists of defining the symbols of second-order arithmetic on N and S. So two distinguished elements of N correspond to the symbols 0 and 1, there are two functions from $N \times N$ to N that correspond to the symbols $+$ and $\times$, and there is a binary relation on N that corresponds to the symbol $<$. As for the symbols $\in$ and $=$, they are interpreted in the usual way.

If a formula is true in a model, then we say that the model **satisfies** the formula. This definition extends to theories, a model **satisfies** a theory if all the axioms of the theory are satisfied by the model. Hence the intended model of the theory Z_2 is $\langle \mathbb{N}, \mathfrak{P}(\mathbb{N})\rangle$, where the symbols have their usual interpretation.

Full models

Among Henkin models, those whose second-order part is exactly $\mathfrak{P}(N)$ are called **full models**. It could seem natural to work only with full models, however, some essential theorems would not hold anymore in that case, e.g. Gödel's completeness theorem, compactness theorem, and Löwenheim-Skolem theorem. Informally, this is because second-order quantification behaves quite differently when it can range over all possible subsets of N.

Moreover, Dedekind's categoricity theorem states that there is only one full model of second-order arithmetic up to isomorphism. Thus, since the theorems we are going to consider are provable in second-order arithmetic, they will be true in every full model, making us unable to prove separation results.

Non-standard models

In first-order logic, there exist models of Peano arithmetic in which there are integers that are not in the set $\omega \coloneqq \{0, 1, 1+1, \dots\}$. Such integers are said to be **non-standard**, and the model itself is called **non-standard**.

One distinctive feature of such integers, making them "infinite" in a sense, is that they are larger than any standard integer. Because of this, some very intuitive properties are actually false in non-standard models. For example, a finite union of finite sets is not necessarily finite. Indeed, first recall that "$X \subseteq \mathbb{N}$ is infinite" is shorthand for $\exists^\infty x, x \in X$, and consider k many finite sets $\{F_i\}_{i<k}$. For each of them there is a threshold $y_i \in \mathbb{N}$ such that $\forall x > y_i, x \notin F_i$, and we wish to find $y \in \mathbb{N}$ such that $\forall x > y, \forall i < k, x \notin F_i$. If k is a standard integer, we can simply consider the threshold $y \coloneqq \max\{y_i : i < k\}$ to be done. This holds even if y is non-standard, because what matters is that $\bigcup_{i<k} F_i$ is finite *from the perspective of the model considered*. However, if k is non-standard, then taking the maximum of the y_i (or simply finding an upper bound) is not necessarily possible.

Definition 1.3.1. A model of second-order arithmetic is an ω**-model** if its first-order part is the set ω of standard integers.

1.3.4 RCA_0

The idea of reverse mathematics is to work above a robust base theory that is as simple as possible while being able to prove theorems of low complexity. The name of this theory is RCA_0, for Recursive Comprehension Axiom, and intuitively it captures computable mathematics. Its axioms are the same as Z_2, but the comprehension scheme is restricted to Δ^0_1 formulas[12], and the induction scheme is restricted to Σ^0_1 formulas. By Theorem 1.2.15, the sets defined in this fashion must be computable, hence the name of the theory. The choice of restriction for the induction scheme can seem more arbitrary, but it is equivalent over $Q + \Delta^0_1$-induction to the assertion that a function can be iterated finitely many times[13], which is a rather tame assumption that nonetheless allows RCA_0 to prove a reasonable amount of statements.

Thanks to RCA_0 we can now compare the strength of theorems. To do so we use the implication over RCA_0, i.e. $\mathsf{RCA}_0 \vdash \mathsf{P} \implies \mathsf{Q}$, which can be interpreted as "theorem P is at least as strong as theorem Q". Observe that, by the completeness theorem, this implication is equivalent to "any model of $\mathsf{RCA}_0 + \mathsf{P}$ is a model of $\mathsf{RCA}_0 + \mathsf{Q}$.

Turing ideals

An ω-model is entirely determined by its second-order part. In the case of RCA_0, the second-order part of its ω-models have a nice characterization in terms of Turing ideals.

Definition 1.3.2. A **Turing ideal** is a non-empty class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ which is

- closed under Turing reduction, i.e. $\forall X \in \mathcal{I}, \forall Y \subseteq \mathbb{N}, Y \leqslant_T X \Rightarrow Y \in \mathcal{I}$
- closed under the join operation, i.e. $\forall X, Y \in \mathcal{I}, X \oplus Y \in \mathcal{I}$

As the name suggests, they form an order ideal for the Turing reduction $\leqslant_T$.

Proposition 1.3.3 (Friedman [Fri74])**.** *An ω-model satisfies RCA_0 if and only if its second-order part is a Turing ideal.*

[12] The notion of Δ^0_1 formula is not syntactic. So to formally write the comprehension scheme for Δ^0_1 formulas, it needs to be modified in the following way: for each Σ^0_1 formula φ and Π^0_1 formula ψ, we have the axiom $\big(\forall x, (\varphi(x) \iff \psi(x))\big) \implies \exists X, \forall y, (y \in X \iff \varphi(y))$

[13] This statement is called PREC, see [HS07, Proposition 6.6] for more information on the equivalence discussed here.

We denote by COMP the ω-model of RCA_0 whose second-order part is the class of all computable sets. It is the smallest model of RCA_0 for inclusion, see [Sim09, Corollary II.1.8]. This further supports the idea that RCA_0 corresponds to "computable mathematics".

1.3.5 The Big Five

The early study of reverse mathematics revealed four subsystems of second-order arithmetic, linearly (and strictly) ordered by the provability relation over RCA_0. Together they are called the "**Big Five**", and they possess a remarkable empirical property: most theorems of "ordinary" mathematics are either provable in RCA_0, or equivalent over RCA_0 to one of the four other subsystems. And so, the burning question is of course

Question 1. Is there any "ordinary" theorem that escapes this phenomenon?

The answer to this question turned out to be positive, and Ramsey's theorem became the first example of a statement escaping the Big Five. More details are given in Section 1.3.8, but before this, we present the subsystems of the Big Five that are relevant to this document.

The subsystem WKL_0

The first subsystem above RCA_0 is WKL_0, it is composed of the axioms of RCA_0, plus an extra statement called weak König's lemma.

Statement 1.3.4 (Weak König's Lemma)**.** Every infinite binary tree T has an infinite path, i.e. $[T] \neq \varnothing$.

The formalization of this statement in second-order arithmetic is written WKL. While it seems to be trivial, it is in fact not provable in RCA_0 (see [Sim09, Example I.8.8]), because there is an infinite binary tree which is computable, but whose paths are all non-computable. The statement WKL is notably equivalent to the Heine-Borel theorem on the closed unit real interval[14] (see [Sim09, §IV.1]), for this reason it is said to capture the notion of compactness.

[14] The theorem states that every covering of the closed unit interval $[0, 1]$ by a sequence of open intervals has a finite subcovering.

There is an equivalent to Turing ideals for WKL_0, named Scott ideals. Recall from Definition 1.2.39 that $X \ll A$ means X is of $\mathsf{PA}(A)$ degree.

Definition 1.3.5. A **Scott ideal** is a Turing ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ that is closed for WKL, i.e. for any infinite tree, computable by some element of $\mathcal{I}$, there is an infinite path in $\mathcal{I}$. Equivalently, a Scott ideal is a Turing ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ such that, $\forall X \in \mathcal{I}, \exists Y \in \mathcal{I}, Y \ll X$.

Unlike RCA_0, the theory WKL_0 does not have a smallest ω-model. In particular, the intersection between all the ω-models of WKL_0 is the class of computable sets, which is not a model of WKL_0. See [Sim09, VIII.2] for more details.

The subsystem ACA_0

The subsystem ACA_0 has the same axioms as RCA_0, but the comprehension axiom scheme is not restricted to Δ^0_1 formulas anymore, it can be applied to any arithmetical formula, hence its name "Arithmetic Comprehension Axiom". It is equivalent over RCA_0 to the statement "every set has a Turing jump" (see [DM22, Corollary 5.6.3]), this characterization is useful to keep in mind in practice. Besides, it gives us an equivalent to Turing ideals for the theory.

Definition 1.3.6. A **jump ideal** is a Turing ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ which is closed for the jump operator, i.e. $\forall X, (X \in \mathcal{I} \implies X' \in \mathcal{I})$.

Like RCA_0, the theory ACA_0 also has a smallest ω-model, namely

$$\mathrm{ARITH} \coloneqq \{X \subseteq \mathbb{N} : \exists n, X \leqslant_T \varnothing^{(n)}\}$$

The other two subsystems

There are two subsystems left to complete our picture of the Big Five hierarchy, see Figure 1.6. Directly above ACA_0 is the subsystem ATR_0, and above the latter is Π^1_1-CA. We will not go into more detail regarding them, as the statements studied in the present document are all provable in ACA_0. For more information about the Big Five, and in particular, these last two subsystems see [Sim09] or [Hir15].

$$\begin{array}{c} \Pi^1_1-\mathsf{CA} \\ \downarrow \\ \mathsf{ATR}_0 \\ \downarrow \\ \mathsf{ACA}_0 \\ \downarrow \\ \mathsf{WKL}_0 \\ \downarrow \\ \mathsf{RCA}_0 \end{array}$$

Figure 1.6: The Big Five. An arrow represents the provability relation, modulo RCA_0.

1.3.6 Problems and reducibilities

In reverse mathematics, many theorems can be seen as **problems**, with **instances** and **solutions**. For example WKL can be seen as a problem whose instances are infinite binary trees, and the solutions of an instance are its infinite paths. More formally, a problem is a Π^1_2 formula of the form $\forall X\big(\Phi(X) \implies \exists Y, \Psi(X,Y)\big)$, where Φ and Ψ are both arithmetical formulas. Thus, an instance is a set X such that $\Phi(X)$, and a solution to X is a set Y such that $\Psi(X,Y)$. We can write $I \in \mathsf{P}$ to signify that I is an instance of the problem P.

Seeing theorems in this fashion offers new ways of studying their relative strength, much alike many-one reducibility in computability theory, instead of just the proof-theoretic implication over RCA_0. These reductions make precise the idea of solving a problem by using our ability to solve another. They all are transitive relations, and Figure 1.7 shows how they relate to one another.

Definition 1.3.7. Let P and Q be two problems.

- P is **computably reducible** to Q, written $\mathsf{P} \leqslant_c \mathsf{Q}$, if every instance I of P computes an instance $\widehat{I}$ of Q such that, for any solution $\widehat{S}$ to $\widehat{I}$, we have $I \oplus \widehat{S}$ computes a solution to I.
- P is **strongly computably reducible** to Q, written $\mathsf{P} \leqslant_{sc} \mathsf{Q}$, if every instance I of P computes an instance $\widehat{I}$ of Q such that any solution to $\widehat{I}$ computes a solution to I.
- P is **strongly omnisciently computably reducible** to Q, written $\mathsf{P} \leqslant_{soc} \mathsf{Q}$, if for every instance I of P, there is an instance $\widehat{I}$ of Q such that any solution to $\widehat{I}$ computes a solution to I.
- P is **Weihrauch reducible** to Q, written $\mathsf{P} \leqslant_W \mathsf{Q}$, if there are Turing functionals Φ and Ψ such that for every instance I of P, we have that Φ^I is an instance of Q, and for any solution $\widehat{S}$ to Φ^I we have that $\Psi^{I \oplus \widehat{S}}$ is a solution to I.
- P is **strongly Weihrauch reducible** to Q, written $\mathsf{P} \leqslant_{sW} \mathsf{Q}$, if there are Turing functionals Φ and Ψ such that for every instance I of P, we have that Φ^I is an instance of Q, and for any solution $\widehat{S}$ to Φ^I we have that $\Psi^{\widehat{S}}$ is a solution to I.

Recall that, for a functional Φ and an oracle A, we denote Φ^A the function $n \mapsto \Phi^A(n)$.

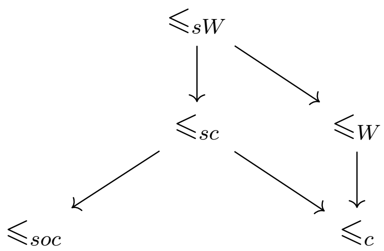

Figure 1.7: Implications between the different reductions. An arrow $\leqslant_a \to \leqslant_b$ means that $\leqslant_a \subset \leqslant_b$, only the implications that hold are represented.

This computability-theoretic approach is related to the proof-theoretic one that we have seen so far. Indeed, if $\mathsf{P} \leqslant \mathsf{Q}$ for any of the above reduction, then every ω-model of Q is an ω-model of P. Also, when restricting ourselves to ω-models, then the implication over RCA_0 is a generalization of the computable reduction, in which it is allowed to have multiple successive applications of the theorem. From this viewpoint, computable reducibility is "resource-sensitive". Besides, Weihrauch reducibility is a uniform version of it, and omniscient reduction is a version where no effectiveness is imposed on the complexity of the instance $\widehat{I}$ of Q.

Each approach has its own interest, as they all reveal different aspects of the relation between two theorems.

1.3.7 Separation and preservation

Separating two problems P and Q, i.e. proving that $\mathsf{RCA}_0 \nvdash \mathsf{P} \implies \mathsf{Q}$, usually consists of expliciting a model $\mathcal{M}$ in which P holds but Q does not, i.e. any instance of P in $\mathcal{M}$ has a solution in $\mathcal{M}$, but there is an instance of Q in $\mathcal{M}$ that has no solution in $\mathcal{M}$. Crafting such a model can prove to be a difficult task, but the notion of preservation can simplify this approach. In particular, it has successfully been used to solve many open questions in reverse mathematics.

Definition 1.3.8. A **weakness property** is a class $\mathcal{W} \subseteq \mathfrak{P}(\mathbb{N})$ downward-closed for Turing reduction, i.e. if $X \in \mathcal{W}$ and $Y \leqslant_T X$ then $Y \in \mathcal{W}$.
Given a weakness property $\mathcal{W}$. A problem P **preserves** $\mathcal{W}$ if, for all $Z \in \mathcal{W}$, any Z-computable instance X of P admits a solution Y such that $Z \oplus Y \in \mathcal{W}$.
Moreover, a problem P **strongly preserves** $\mathcal{W}$ if any instance X of P admits a solution $Y \in \mathcal{W}$.

Classic examples of weakness properties are:
- the class of low sets
- the class of computably dominated sets
- cone avoidance (the complement of the cone above some non-computable set)
- the class of arithmetical sets

Proposition 1.3.9. *If a problem* Q *preserves a weakness property* $\mathcal{W}$*, and* P *does not, then there is a model of* P *which is not a model of* Q*. Thus* $\mathsf{RCA}_0 \nvdash \mathsf{P} \implies \mathsf{Q}$.

For a proof of this claim see [MP22, Corollaire 24.1.12].

The basis theorems we have seen in Section 1.2.10 can be rephrased in terms of preservation. Hence, WKL preserves lowness, computable domination, and cone avoidance. Indeed, for the case of lowness, let Z be a low set, and let T be an infinite Z-computable binary tree. By the low basis theorem relativized to Z, there is $P \in [T]$ such that $(Z \oplus P)' \leqslant_T Z'$. Since Z is low, i.e. $Z' \leqslant_T \varnothing'$, then $Z \oplus P$ is also low, and so WKL preserves lowness. The proof for the other properties is similar.

1.3.8 Ramsey's theorem

This section offers a short historical survey on Ramsey's theorem in reverse mathematics, a result in combinatorics which received a particular interest, as it provided the first example of a "natural" statement escaping the big five phenomenon, and consequently fostered the development of different techniques in the field. We shall present the theorem itself and the main results related to its study, many have been reformulated to fit with current notations. Good references on the subject are [Hir15], [MP22] and [DM22].

1928: Ramsey's initial statement

In 1928, the 25-year-old mathematician **Frank Ramsey** published a paper titled *On a problem of formal logic* [Ram30]. According to the abstract, he was "primarily concerned with a special case of one of the leading problems of mathematical logic" (the Entscheidungsproblem), but by doing so, he stumbled upon some combinatorial theorems which were interesting on their own. The theorem he labeled A is what we now call Ramsey's theorem.[15]

Before being able to state Ramsey's theorem, we need some notation and definitions.

Notation 1.3.10. Given a set $X \subseteq \mathbb{N}$ and any integer n, we define

$$[X]^n \coloneqq \{F \subseteq X : \operatorname{card}(F) = n\}$$

The set $[X]^n$ is in one-to-one correspondence with the set $\{(x_0, \ldots, x_{n-1}) \in X^n : x_0 < \ldots < x_{n-1}\}$. Hence, its elements are referred to as "the n-tuples of X", or simply "the n-tuples" if $X = \mathbb{N}$.

[15] To be more precise, Ramsey proved three theorems, labeled A, B, and C. Theorem B is actually what is generally referred to as Ramsey's theorem nowadays, and theorem C is just an equivalent form of theorem B. Theorem A is an infinite version of theorem B, which is why it is the preferred version in areas such as set theory and reverse mathematics.

Figure 1.8: Frank Plumpton Ramsey

Definition 1.3.11. Given a set X, and two integers n and k, a k-**coloring** of the n-tuples is a function from $[X]^n$ to $\{0, \dots, k-1\}$.

We simply say "coloring" when the values of n and k can be inferred or are irrelevant. Furthermore, we write $f(x_0, \dots, x_{n-1})$ instead of $f(\{x_0, \dots, x_{n-1}\})$ and assume $x_0 < \dots < x_{n-1}$.

Definition 1.3.12. Given a coloring $f : [X]^n \to k$, a set $H \subseteq X$ is f-**homogeneous** if f is constant on $[H]^n$, i.e. $\operatorname{card}\big(f\big([H]^n\big)\big) = 1$.

Finally, we can state the theorem.

Statement 1.3.13 (Ramsey's theorem (RT^n_k))**.** For all coloring $f : [\mathbb{N}]^n \to k$, there exists an infinite f-homogeneous set.

The formalization of this statement in second-order arithmetic is written RT^n_k. To see proofs of this theorem, the author recommends [Hir15, Section 6.1].

For $n = 1$, Ramsey's theorem corresponds to the infinite pigeonhole principle, i.e. if infinitely many objects are colored in finitely many colors, then there is a color that corresponds to infinitely many objects. For $n > 1$, the structure of a coloring is more complicated, and Ramsey's theorem can be seen as a way to find some regularity in any structure that is big enough.

Folklore

In computability theory, some properties of Ramsey's theorem were probably well known and part of the folklore, we begin by reviewing them.

Computationally speaking, RT^1_k is quite weak, as any computable instance has a computable solution. Indeed, an algorithm can simply consider a color it believes is used infinitely many times, and then select all the elements of that color, in order to build an infinite homogeneous set. Throughout the k different algorithms possible, there is at least one that gives the correct answer. In other words:

Proposition 1.3.14. *For any $k \in \mathbb{N}$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^1_k$*

Another property is that, the bigger the value of n, the more difficult it is to solve RT^n_k, in other words, instances of RT^n_k are easier to solve than instances of RT^{n+1}_k, i.e.

Proposition 1.3.15. *For all $n, k \in \mathbb{N}$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^{n+1}_k \implies \mathsf{RT}^n_k$*

Proof. We can encode a given coloring $f : [\mathbb{N}]^n \to k$ into a coloring $\widetilde{f} : [\mathbb{N}]^{n+1} \to k$ with a dummy variable, i.e.

$$\widetilde{f} : x_0 < \ldots < x_n \mapsto f(x_0, \ldots, x_{n-1})$$

By RT^{n+1}_k there is an infinite $\widetilde{f}$-homogeneous set $H \subseteq \mathbb{N}$, and by definition of $\widetilde{f}$ it is also f-homogeneous. □

Finally, we can restrict our study to 2-colorings, instead of k-colorings for $k \geqslant 2$, without any loss of generality, because instances of RT^n_{k+1} can be solved by using multiple instances of RT^n_k. This is called a **color-blindness argument**, and it requires the following lemma. In essence, it shows that it is equivalent to use any infinite set $X \subseteq \mathbb{N}$ instead of $\mathbb{N}$, in the statement of Ramsey's theorem.

Lemma 1.3.16. *Let P be the statement "for any infinite set $X \subseteq \mathbb{N}$, and for any coloring $f : [X]^n \to k$, there is an infinite f-homogeneous set". Then* $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \implies P$.

Proof. Let $X \coloneqq \{x_0 < x_1 < \ldots\}$ be an infinite set and $f : [X]^n \to k$ be an instance of P. By Σ^0_1 induction, define the bijection $h : n \mapsto x_n$ between $\mathbb{N}$ and X. Then, by RT^n_k, there is an infinite set $H \subseteq \mathbb{N}$ that is homogeneous for the coloring $\widetilde{f} : a_0, \ldots a_{n-1} \mapsto f(h(a_0), \ldots, h(a_{n-1}))$. Finally, the infinite set $\{g(a) : a \in H\}$ is Δ^0_1 and homogeneous of f. □

Proposition 1.3.17. *For all $k \geqslant 1$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_2 \implies \mathsf{RT}^n_{k+1}$*

Proof. We proceed by induction on k, for $k = 1$ the result is trivial. Given $f : [\mathbb{N}]^n \to k+1$ consider

$$\widehat{f} : [\mathbb{N}]^n \to k$$
$$\vec{x} \mapsto \begin{cases} k-1 & \text{if } f(\vec{x}) = k \\ f(\vec{x}) & \text{otherwise} \end{cases}$$

Then, by induction hypothesis, there is an infinite set $H \subseteq \mathbb{N}$ homogeneous for $\widehat{f}$. If the color is not k, then it is also f-homogeneous. Otherwise, consider $f\restriction_{[H]^n}$, it is a 2-coloring, and by the previous lemma, there is an infinite homogeneous set G, from which we can deduce an infinite f-homogeneous set. □

1966: First steps in computability theory with Specker

The study of Ramsey's theorem from a computational point of view started in the mid-'60s, before the existence of reverse mathematics. The first natural question to ask is probably the following:

Question 2. Does a computable instance of RT^n_k always have a computable solution?

We have already seen the answer for the case $n = 1$. A negative answer for the case $n \geqslant 2$ was brought by **Specker**. He constructed a computable instance of RT^2_2 with no c.e. solution.

Proposition 1.3.18 (Specker)**.** *If $n \geqslant 2$ and $k \geqslant 2$, then* $\mathsf{RCA}_0 \nvdash \mathsf{RT}^n_k$.

This result was published in 1971 in a paper titled *Ramsey's theorem does not hold in recursive set theory* [Spe71], but was presented during a talk in Manchester as early as 1966 (see footnote in Specker's paper).

1972: Jockusch's first results

After Specker's result, the next natural question to ask is

Question 3. How complex can a solution to a computable instance of RT^n_k be?

In 1972, **Jockusch** answered the question when he published the first paper in which Ramsey's theorem was being thoroughly investigated, titled *Ramsey's Theorem and Recursion Theory* [Joc72b]. He established optimal bounds on the complexity of solutions for computable instances of RT^n_k, for $n \geqslant 2$.

Theorem 1.3.19 (Jockusch [Joc72b, Theorem 5.1 and 5.5])**.** *Let $n \geqslant 2$ and $k \geqslant 1$. There is a computable instance of RT^n_2 with no Σ^0_n solution. Every computable instance of RT^n_k admits a Π^0_n solution.*

Corollary 1.3.20. *For all $n, k \in \mathbb{N}$, $\mathsf{ACA}_0 \vdash \mathsf{RT}^n_k$*

Moreover, he proved that RT^3_2 has an instance such that any of its solutions computes $\varnothing'$ [Joc72b, Theorem 5.7]. This result allows for the construction of a jump ideal, and leads to the following result, by starting from a countable model and then adding a solution to any instance of RT^3_2 in the model.

Theorem 1.3.21 (Jockusch)**.** *For all $n > 3$ and all $k > 2$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \iff \mathsf{ACA}_0$*

In addition to these results, Jockusch published the same year, with **Soare**, another paper titled Π^0_1 *classes and degrees of theories* [JS72a], in which they proved the low basis theorem. With that extra tool, it becomes possible to prove the following.

Theorem 1.3.22. $\mathsf{WKL}_0 \nvdash \mathsf{RT}^2_2$

Indeed, we can construct a model of WKL_0 which is not a model of RT^2_2, i.e. in the model there is an instance of RT^2_2 that has no solution in the model. On one hand, by the low basis theorem it is possible to construct a model with only low solutions. To do so, consider a countable model of RCA_0. For any instance of WKL in the model, there is an infinite low path by the low basis theorem. This path is added to the model along with all the sets computable from it, so the model remains downward closed for Turing reduction. This procedure can be repeated for every instance of WKL in the model, since the latter is countable. On the other hand, by Jockusch's theorem, there is a computable instance of RT^n_2 with no Σ^0_n

solution, in particular, using Post's theorem, it has no $\varnothing'$-computable solution, this means that no low set can compute any solution.

So only the case $n = 2$ remains open to finish the study of Ramsey's theorem, with one big question.

Question 4. Does RT^2_2 imply ACA_0 above RCA_0?

Answer in Theorem 1.3.27.

1987: Hirst's thesis and the bounding principle

In his thesis [Hir87, Chapter 6], **Hirst** proved some interesting results regarding Ramsey's theorem, notably some related to the **bounding principle**.

Statement 1.3.23 (Bounding principle)**.**

$$\forall a, \big((\forall x < a, \exists y, \varphi(x, y) \implies \exists b, \forall x < a, \exists y < b, \varphi(x, y)\big)$$

where φ is a formula.

In particular, if φ is a function, then the statement can be interpreted as "the image of a bounded set by a function is bounded". The formalization of the bounding principle in second-order arithmetic is denoted $\mathsf{B}\Gamma$, where Γ is a class of formulas from which φ is taken. What makes this statement non-obvious is the fact that a can be a non-standard integer, in which case finding a suitable b that bounds the values of y becomes more difficult than in the standard case. In particular, the statement $\mathsf{B}\Sigma^0_2$ is often encountered in reverse mathematics.[16] Firstly because there is a link between the induction scheme and the bounding scheme. Namely, for any $n \in \mathbb{N}$, $\mathsf{I}\Sigma^0_{n+1} \implies \mathsf{B}\Sigma^0_{n+1} \implies \mathsf{I}\Sigma^0_n$, see [PK78, Theorem A]. Secondly, because it is linked to Ramsey's theorem in the following way.

Notation 1.3.24. For any n, let $\mathsf{RT}^n_{<\infty}$ (sometimes simply written RT^n) denote the statement $\forall k, \mathsf{RT}^n_k$.

[16] Note that $\mathsf{B}\Sigma^0_2$ is equivalent to $\mathsf{B}\Pi^0_1$. Indeed, the existential quantification in the Σ^0_2 formula can be merged with the existential quantification in the bounding principle. More generally, $\mathsf{B}\Sigma^0_{n+1}$ is equivalent to $\mathsf{B}\Pi^0_n$, for any $n \geqslant 1$.

Proposition 1.3.25 ([Hir87, Theorem 6.4]). $\mathsf{RCA}_0 \vdash \mathsf{RT}^1_{<\infty} \iff \mathsf{B}\Sigma^0_2$

This characterization is used by Hirst to show that $\mathsf{RT}^1_{<\infty}$ cannot be proven in RCA_0 nor WKL_0, see [PK78, Corollary 6.5]. Ultimately leading to a new proof that $\mathsf{WKL}_0 \nvdash \mathsf{RT}^2_2$ based on the following fact.

Proposition 1.3.26 ([PK78, Theorem 6.8]). $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \implies \mathsf{RT}^1_{<\infty}$

1995: Seetapun's theorem

An answer to Question 4 came in 1995, when **Seetapun**, who was a student of **Slaman** at the time, proved the following result in their paper *On the strength of Ramsey's theorem* [SS95, Theorem 3.1].

Theorem 1.3.27 (Seetapun). $\mathsf{WKL}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{ACA}_0$

To prove this, Seetapun constructed a model of WKL_0 (more precisely, a countable ω-model, whose second-order part is a Scott ideal S), which is a model of RT^2_2 but not a model of ACA_0, i.e. every instance of RT^2_2 in S has a solution in S, but there is a set $Z \in S$ whose jump Z' is not in S. To achieve this, it is necessary to have the following result called **cone avoidance of** RT^2_2.

Theorem 1.3.28 (Seetapun [SS95, Theorem 2.1]). *Let $Z \subseteq \mathbb{N}$. For any Z-computable coloring $f : [\mathbb{N}]^2 \to 2$, if $(C_i)_{i\in\mathbb{N}}$ is a sequence of sets such that $\forall i, C_i \not\leqslant_T Z$, then there is an infinite f-homogeneous set $H \subseteq \mathbb{N}$ such that $\forall i, C_i \not\leqslant_T H$.*

From there, to prove Theorem 1.3.27, first consider COMP. It is a model of RCA_0 that contains $\varnothing$ but not its jump $\varnothing'$. Then consider some instance of RT^2_2 in the model. By using Theorem 1.3.28, there is a solution H that avoids the cone above $\varnothing'$. This solution is added to the model, and to ensure the model is downward closed for $\leqslant_T$, we add all the sets that are H-computable. Due to the property of H, we know $\varnothing'$ has not been added at this step. Finally, since the model considered is countable, this procedure can be done for all the instances of RT^2_2 in the model.

So RT^2_2 is strictly below ACA_0, the remaining big question now becomes

Question 5. Does RT^2_2 imply WKL over RCA_0?

Answer in Theorem 1.3.40.

2001: CJS paper and the $\mathsf{SRT}^2_2 + \mathsf{COH}$ decomposition

After the publication of [SS95], **Cholak**, **Jockusch** and **Slaman** decided to investigate the remaining question. This resulted in a paper titled *On the strength of Ramsey's theorem for pairs* [CJS01], colloquially known after the authors initials "CJS", published in 2001, and containing numerous results regarding RT^2_2, many of which were probably shared with the community before publication. It also shaped the future studies of RT^2_2 by asking questions and proposing different approaches.

A notable contribution of the paper (see §7) was the decomposition of RT^2_2 in two separate statements. The first statement is called COH and corresponds to a property called **cohesiveness**. It can be seen as a sequential version of RT^1_2, indeed it says that for every infinite sequence of 2-colorings of the integers, there is an infinite set that is almost homogeneous for all the colorings.

Definition 1.3.29. A set A is **almost included** in a set B, denoted $A \subseteq^* B$, if $B - A$ is finite, in other words $\forall^\infty x \in A, x \in B$. Given an infinite sequence of sets $\vec{R}$, a set C is **$\vec{R}$-cohesive** if, for any $i \in \mathbb{N}$, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$.

Statement 1.3.30 (COH)**.** For any infinite sequence of sets $\vec{R}$, there is an infinite $\vec{R}$-cohesive set.

This statement is especially useful because it creates a bridge between computable instances of RT^2_k and Δ^0_2 instances of RT^1_k. Indeed, consider a computable coloring $f : [\mathbb{N}]^2 \to k$ and define the sequence of sets $\vec{R} := (R_{x,i})_{x\in\mathbb{N},i<k}$, where $R_{x,i} := \{y \in \mathbb{N} : f(x, y) = i\}$. If C is an infinite $\vec{R}$-cohesive set, then $\lim_{y\in C} f(x, y)$ exists for all $x \in C$. Thus we can define the coloring $g : C \to k$ such that $g(x) := \lim_{y\in C} f(x, y)$. It is $(C \oplus f)'$-computable because, for any $x \in C$, we can search for a threshold $s \in \mathbb{N}$ and a color $i < k$ by using the halting set to know whether or not $\forall y > s \in C, f(x, y) = i$. Since the limit of f exists, we are guaranteed to find s and i, in which case we then define $g(x) := i$. For any $Y \subseteq C$ solution to g, we have that $Y \oplus f$ computes a solution to f. Indeed, we can proceed by induction. First define $x_0 \coloneqq \min Y$. Then, suppose $\{x_0, \dots, x_{n-1}\}$ is f-homogeneous for the color $i < k$, and define x_n to be the smallest element of Y that is larger than x_{n-1} and such that $\forall j < n, f(x_j, x_n) = i$.

The second statement is a restriction of Ramsey's theorem for stable colorings.

Definition 1.3.31. For any n, a coloring $f : [\mathbb{N}]^{n+1} \to k$ is **stable** if, for every $\vec{x} \in [\mathbb{N}]^n$, $\lim_y f(\vec{x}, y)$ exists, i.e. $\forall \vec{x} \in [\mathbb{N}]^n, \exists i < k, \forall^\infty y, f(\vec{x}, y) = i$.

Statement 1.3.32. (Stable Ramsey's theorem SRT^n_k) For any stable coloring $f : [\mathbb{N}]^n \to k$, there is an infinite f-homogeneous set.

Hence the decomposition of RT^2_2 can be formally written as follows.[17]

Proposition 1.3.33 ([CJS01, Lemma 7.11], [Mil04, Appendix A])**.**
$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \iff \left(\mathsf{SRT}^2_2 + \mathsf{COH}\right)$

Naturally, the question arising from this fact is "Is $\mathsf{SRT}^2_2 + \mathsf{COH}$ a strict decomposition of RT^2_2?". More precisely:

Question 6. Does $\mathsf{RCA}_0 \vdash \mathsf{COH} \implies \mathsf{RT}^2_2$ hold?

Answer in Theorem 1.3.39.

Question 7. Does $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \implies \mathsf{RT}^2_2$ hold?

Partial answer in Section 1.3.8, completed by Section 1.3.8.

A first approach proposed in [CJS01, Question 13.9] to solve Question 7 relies on another question.

Question 8. Does every computable instance of SRT^2_2 have a low solution?

If the answer is yes, a similar argument as the one used for Theorem 1.3.22 would prove that $\mathsf{RCA}_0 \nvdash \mathsf{SRT}^2_2 \implies \mathsf{RT}^2_2$. That is to say, on one hand it would

[17] The proof of $\mathsf{RT}^2_2 \implies \mathsf{COH}$ proposed in [CJS01] turned out to be erroneous, indeed it unknowingly used $\mathsf{I}\Sigma^0_2$, which is not provable in RCA_0. Mileti later provided a corrected proof in his PhD thesis.

be possible to make a model $\mathcal{M}$ of SRT^2_2 with only low solutions, and on the other hand there is an instance of RT^2_2 with no Δ^0_2 solution, meaning that $\mathcal{M}$ is not a model of RT^2_2. Unfortunately, a negative answer to that question was brought by **Downey**, **Hirschfeldt**, **Lempp** and **Solomon** in their 2001 article *A Δ^0_2 Set with No Infinite Low Subset in Either It or Its Complement* [DHLS01]. The proof uses a rather complex **infinite injury priority argument**; to learn more about this type of proof see [Lem] or [DH10, 2.11-2.14].

Before continuing, we state an important theorem of [CJS01] that gives another insight regarding the complexity of the solutions of Ramsey's theorem for pairs.

Theorem 1.3.34 ([CJS01, Theorem 3.1])**.** *For any computable instance of RT^2_k, there is a low$_2$ solution, i.e. an infinite homogeneous set H such that $H'' \leqslant_T \varnothing''$.*

2005 and 2007: the ADS + EM decomposition

In 2005, another decomposition was unveiled by **Bovykin** and **Weiermann** in their paper *The strength of infinitary Ramseyan principles can be accessed by their densities* [BW17]. They proved that RT^2_2 is equivalent to the combination of two statements called CAC and EM (see [BW17, Theorem 8]). The former will be studied in greater detail in Chapter 2, for a definition see Statement 2.1.1. We now define the latter.

Definition 1.3.35. A coloring $f : [\mathbb{N}]^2 \to 2$ is **transitive** for a set $H \subseteq \mathbb{N}$ if for any color $i < 2$, and any $x<y<z \in H$ we have

$$f(x, y) = f(y, z) = i \implies f(x, z) = i$$

Accordingly, the set H is said to be **transitive** for f. If the context is clear, or if $H = \mathbb{N}$, then we can simply say "transitive" in both cases.

Statement 1.3.36 (Erdős-Moser (EM))**.** For all coloring $f : [\mathbb{N}]^2 \to 2$, there exists an infinite transitive set.

Moreover, according to their paper, they were informed by **Montalbán** that CAC can be replaced with a weaker principle called ADS.

Statement 1.3.37 (Ascending Descending Sequence)**.** Any infinite linear order has an infinite sequence that is either ascending or descending.

Theorem 1.3.38. $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \iff (\mathsf{ADS} + \mathsf{EM})$

These statements, along with others, have then been studied more extensively by **Hirschfeldt** and **Shore** in *Combinatorial Principles Weaker than Ramsey's Theorem for Pairs* [HS07], written in 2007. Their paper gives a more precise picture of the hierarchy of the statements below RT^2_2. However, they did not answer the natural question that arose from this new decomposition.

Question 9. Is $\mathsf{ADS} + \mathsf{EM}$ a strict decomposition of RT^2_2?

Answer in Theorem 1.3.42.

2008: COH does not imply RT^2_2

Around 2008, a negative answer to Question 6 was provided by **Hirschfeldt**, **Jockusch**, **Kjos-Hanssen**, **Lempp** and **Slaman** in their paper *The Strength of Some Combinatorial Principles Related to Ramsey's Theorem for Pairs* [HJKH+08].

Theorem 1.3.39. $\mathsf{RCA}_0 \nvdash \mathsf{COH} \implies \mathsf{RT}^2_2$

To prove this result, they showed that SRT^2_2 implies a statement called DNR (see [HJKH+08, Theorem 2.4]), whereas $\mathsf{RCA}_0 + \mathsf{COH}$ does not prove this statement (see [HJKH+08, Theorem 3.7]), hence $\mathsf{RCA}_0 + \mathsf{COH}$ does not prove SRT^2_2. In the paper, they also asked the following question, as an alternative approach to prove Question 7. As of today, it is still open.

Open question 1 ([HJKH+08, Question 1.1]). Let A be Δ^0_2. Is there an infinite subset in either A or $\overline{A}$, that is both Δ^0_2 and low$_2$?

June 2012: Liu's theorem

In 2012, a negative answer was brought to Question 5 by **Liu**.

Theorem 1.3.40 (Liu). $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2 \implies \mathsf{WKL}$

This came as a big surprise since at the time Liu was only an undergraduate student unknown to the rest of the community. He wrote his proof in a paper called "RT^2_2 does not imply WKL" [Liu12] and sent it to the Journal of Symbolic Logic for publication. A simplified version of the proof was later given in Hirschfeldt's book *Slicing the Truth* [Hir15, Lagniappe]. In his article, Liu proved that non-PA degrees are strongly preserved by RT^1_k. This implies that RT^2_k preserves non-PA degrees, because of the bridge we saw earlier between computable instances of RT^2_k and Δ^0_2 instances of RT^1_k.

Theorem 1.3.41 (Liu [Liu12, Theorem 1.5]). *For any set C not of PA-degree and any set A. There exists an infinite subset G of A or $\overline{A}$, such that $G \oplus C$ is also not of PA-degree.*

Once again this allows for the construction of a model of RT^2_2 that is not a model of WKL_0, see [Liu12, Corollary 1.6] for the detailed construction.

Hence, after four decades of research, the status of RT^n_k was known for every n and k. In particular, RT^2_2 became the first example of a "natural" theorem proven to escape the Big Five phenomenon. The most pressing questions remaining were regarding the different decompositions of RT^2_2.

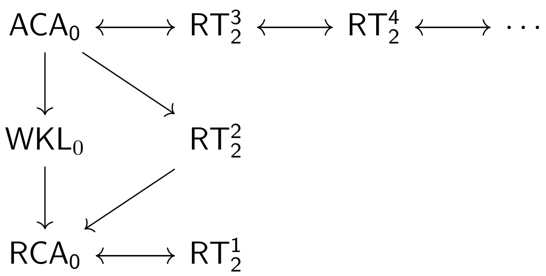

Figure 1.9: Ramsey's theorem in the Big Five hierarchy

Decembre 2012: computable and Weihrauch reduction

At the end of 2012, new notions of reducibility between principles were being investigated. The idea was to cast a new light on known statements to learn more about them, and hopefully bring answers to open questions. On one hand, **Dzhafarov** defined computable reducibility and its strong counterpart in *Cohesive avoidance and arithmetical sets* [Dzh12].[18] On the other hand **Dorais**, **Dzhafarov**, **Hirst**, **Mileti**, and **Shafer** investigated Weihrauch reducibility and its strong counterpart in their paper *On uniform relationships between combinatorial problems* [DDH+16].[19]

Note that these new reducibilities brought new questions regarding Ramsey's theorem. For example, in line with [DDH+16, Question 7.1]:

Question 10. Does $\mathsf{RT}^n_{k+1} \leqslant_c \mathsf{RT}^n_k$ hold?

Answer in Theorem 1.3.44.

Indeed, we have seen with Proposition 1.3.17 that the number of colors is not relevant to the study of Ramsey's theorem when it is standard. Nonetheless, the new reductions are "resource-sensitive", i.e. they do not allow multiple applications of the theorem in the reduction. So the method used in Proposition 1.3.17 does not hold anymore. In [DDH+16, Theorem 3.1], a partial answer is given, the authors proved that $\forall 2 \leqslant j < k, \mathsf{RT}^n_k \not\leqslant_{sW} \mathsf{RT}^n_j$.

2013: $\mathsf{ADS} + \mathsf{EM}$ is a strict decomposition of RT^2_2

In 2013, **Lerman**, **Solomon** and **Towsner** answered Question 9 in a paper titled *Separating principles below Ramsey's Theorem for Pairs* [LST13]

Theorem 1.3.42. *The decomposition of* RT^2_2 *into* $\mathsf{ADS} + \mathsf{EM}$ *is strict.*

Specifically, they used iterated forcing to prove that ADS is not equivalent to CAC, and that EM does not imply RT^2_2. Moreover, in 2015, **Patey** wrote a simpler proof in a paper called *Iterative forcing and hyperimmunity in reverse mathematics* [Pat15].

[18] This paper was published in 2014 under a slightly different name *Cohesive avoidance and strong reductions* [Dzh14]

[19] The paper was only published in 2016, but the first online version is from 2012. Moreover, in that first version, the authors actually define what they call "uniform reducibility". Only later did they learn they had rediscovered Weihrauch reducibility.

2014: partial answer for SRT^2_2

In 2014, **Chong**, **Slaman**, and **Yang** partially solved our only remaining question, Question 7, in their paper *The metamathematics of Stable Ramsey's Theorem for Pairs* [CSY14]. They proved that SRT^2_2 does not imply RT^2_2 by constructing a model of $\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ whose first-order part is non-standard and in which every set is low [CSY14, Theorem 2.2]. Indeed, the proof mentioned earlier dismissing the approach of Question 8 relies on $\mathsf{I}\Sigma^0_2$ to work. However, by forcing, a model with a non-standard first-order part can verify the negation of $\mathsf{I}\Sigma^0_2$, allowing us to build a model in which SRT^2_2 always has a low solution. To fully answer Question 7 we now need to know if this result also applies to ω-models.

Question 11. Is there an ω-model of SRT^2_2 that is not a model of RT^2_2?

Answer in Section 1.3.8.

2014 to 2016: results with computable and Weihrauch reduction

In 2014, **Dzhafarov** used the new reductions uncovered a few years prior, to investigate SRT^2_2 and COH. This resulted in a paper titled *Strong Reductions Between Combinatorial Principles* [Dzh16], where he notably proved the following result.

Proposition 1.3.43. $\mathsf{COH} \not\leqslant_{sc} \mathsf{SRT}^2_2$

Since strong Weihrauch reducibility implies strong computable reducibility, the above result is also valid for strong Weihrauch reduction.

In 2015, **Hirschfeldt** and **Jockusch** submitted a paper, titled *On notions of computability-theoretic reduction between Π^1_2 principles* [HJ16], in which the new reductions are used. Like in [CJS01], many results were proven, notably they showed that, for $n \geqslant 3$, there exists a computable instance of RT^n_2 such that every solution is of PA degree over $\varnothing^{(n-2)}$ [HJ16, Corollary 2.2].

Later, in 2016, the authors were informed that some results in their paper were also obtained independently by **Patey** in [Pat16c], as well as **Rakotoniaina** during his PhD under the supervision of **Brattka**. In particular, they all proved that $\forall n \geqslant 2, \forall k > \ell \geqslant 2, \mathsf{RT}^n_k \not\leqslant_W \mathsf{RT}^n_\ell$ [HJ16, Theorem 3.3], which improved the result of 2012. However, **Patey** had proven a stronger result that answered Question 10.

Theorem 1.3.44 ([Pat16c, Corollary 3.14]). *For every $n \geqslant 2$ and every $k > \ell \geqslant 2$, we have* $\mathsf{SRT}^n_k \not\leqslant_c \mathsf{RT}^n_\ell$

2019: final answer for SRT^2_2

Finally, in 2019, roughly twenty years after the question was asked, **Monin** and **Patey** confirmed that $\mathsf{SRT}^2_2 + \mathsf{COH}$ is a strict decomposition, even for ω-models. Their result was published in 2021, in a paper called SRT^2_2 *does not imply* RT^2_2 *in omega-models* [MP21], thus answering Question 11.

Disclaimer and open questions

The above presentation is only an approximate story where many details, open questions, and theorems had to be excluded for conciseness. For a list of open questions[20] see [Mon11] and [Pat16a].

[20]Some of them may already be answered now, due to the age of the documents.

CHAPTER 2

CAC FOR TREES

2.1 Introduction

In this chapter, we study the computability-theoretic strength of the statement CAC for trees, which is a variation on the well-studied chain-antichain theorem (CAC). It turns out CAC for trees has different characterizations, making it a robust notion, suitable for future studies in reverse mathematics.

The study of CAC for trees started with the study of Ramsey-like theorems for 3-variable forbidden patterns. The attempt to prove Corollary 2.6.13 naturally led to the study of the SHER principle, already defined by Dorais and al. [DDH$^+$16]. Thanks to multiple personal communications with François Dorais, we realized that the SHER principle is closely related to trees, and more precisely, equivalent to the chain-antichain principle for trees, a principle studied by Binns et al. in [BKHL$^+$14]. We later realized that SHER is also equivalent to $\mathsf{TAC} + \mathsf{B}\Sigma^0_2$, where TAC is an antichain principle for completely branching c.e. trees, defined by Conidis [Con]. Some of the results are therefore independent rediscoveries of some theorems from [BKHL$^+$14, Con], but in a more unified setting.

2.1.1 A chain-antichain theorem for trees

Among the consequences of Ramsey's theorem for pairs, the chain-antichain theorem received a particular focus in reverse mathematics.

Statement 2.1.1 (Chain AntiChain (CAC))**.** Every infinite partial order has either an infinite chain or an infinite antichain.

CAC was first studied in [HS07] by Hirschfeldt and Shore, following a question raised by Cholak, Jockusch and Slaman in [CJS01, Question 13.8] asking whether or not $\mathsf{CAC} \implies \mathsf{RT}^2_2$ over RCA_0, for which they proved the answer is negative (Corollary 3.12). The reciprocal $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \implies \mathsf{CAC}$ is easier to obtain, by defining a coloring such that $\{x, y\}$ has color 1 if its elements are comparable, and 0 otherwise. Any homogeneous set for this coloring is either a chain or an antichain, depending on its color.

In this chapter, we focus on the special case where the order is the predecessor relation $\prec$ on a tree.

Statement 2.1.2 (CAC for trees)**.** Every infinite (binary) subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path or an infinite antichain.

This statement was first introduced by Binns et al. in [BKHL+14], where the authors showed that every infinite computable tree must have either an infinite computable chain or an infinite Π^0_1 antichain. Furthermore, they showed that these bounds are optimal, by constructing an infinite computable tree that has no infinite Σ^0_1 chain or antichain. They also showed that $\mathsf{WKL}_0 \nvdash \mathsf{CAC}$ for binary trees, [BKHL+14, Corollary 6.5].

2.1.2 Ramsey-like statements

In [Pat19], Patey identified a formal class of theorems, encompassing several statements surrounding Ramsey's theorem. Indeed, many of them are of the form "for every coloring $f : [\mathbb{N}]^n \to k$ avoiding some set of forbidden patterns, there exists an infinite set $H \subseteq \mathbb{N}$ avoiding some other set of forbidden patterns (relative to f)". Such statements are called **Ramsey-like theorems**.

For example, recall that EM asserts that "for any coloring $f : [\mathbb{N}]^2 \to 2$, there exists an infinite set $H \subseteq \mathbb{N}$ which is transitive for f", i.e. $\forall i < 2, \forall x{<}y{<}z \in H, f(x, y) = i \land f(y, z) = i \implies f(x, z) = i$. In other terms, we want H to avoid the patterns, $f(x, y) = i \land f(y, z) = i \land f(x, z) = 1 - i$ for any $i < 2$, that would make it not transitive for f. Another example is ADS, which is equivalent over RCA_0 to the statement "for any transitive coloring $f : [\mathbb{N}]^2 \to 2$ (i.e. avoiding certain patterns), there exists an infinite set $H \subseteq \mathbb{N}$ which is f-homogeneous" (see [HS07, Theorem 5.3]). With these formulations, the equivalence between RT^2_2 and $\mathsf{EM} + \mathsf{ADS}$ over RCA_0 is very clear, since EM takes any coloring and "turns it into" a transitive one, and ADS takes any transitive coloring and finds an infinite set which is homogeneous for it.

2.1.3 Forbidden patterns on 3 variables

Forbidden patterns on 3 variables and 2 colors are generated by the following three basic patterns:

(1) $f(x, y) = i \wedge f(y, z) = i \wedge f(x, z) = 1 - i$
(2) $f(x, y) = i \wedge f(y, z) = 1 - i \wedge f(x, z) = i$
(3) $f(x, y) = 1 - i \wedge f(y, z) = i \wedge f(x, z) = i$

Avoiding them respectively leads to **transitivity**, **semi-ancestry**, and **semi-heredity** (for the color i). Each of them generates two ramsey-like statements, one restricting the input coloring, and one restricting the output infinite set, namely "for any 2-coloring of pairs avoiding the forbidden pattern, there exists an infinite homogeneous set" and "for any 2-coloring of pairs, there exists an infinite set $H \subseteq \mathbb{N}$ which avoids the forbidden pattern". We now survey the known results about these three patterns.

Transitivity. The statement "for any 2-coloring of pairs, there exists an infinite set which is transitive for some color" is a weaker version of EM. The Erdös-Moser theorem was proven to be strictly weaker than Ramsey's theorem for pairs over RCA_0 by Lerman, Solomon, and Towsner [LST13, Corollary 1.16]. On the other hand, the statement "for any 2-coloring of pairs which is transitive for some color, there exists an infinite homogeneous set" is equivalent to CAC (see [HS07, Theorem 5.2]), which is also known to be strictly weaker than RT^2_2 over RCA_0 (see Hirschfeldt and Shore [HS07, Corollary 3.12]).

Semi-ancestry. The statement "for any 2-coloring of pairs which has semi-ancestry for some color, there exists an infinite homogeneous set" is a consequence of the statement STRIV, defined by Dorais et al. [DDH+16, Statement 5.12]), because a 2-coloring is semi-trivial if and only if it has semi-ancestry. And STRIV itself is equivalent to $\mathsf{RT}^1_{<\infty}$ (see the remark below its definition). The statement "for any 2-coloring of pairs, there exists an infinite set which has semi-ancestry for some color" is equivalent to RT^2_2 (see Proposition 2.6.11).

Semi-heredity. The statement "for any 2-coloring of pairs which is semi-hereditary for some color, there exists an infinite homogeneous set" is the statement SHER, which was first introduced by Dorais et al. [DDH+16, Statement 5.11]. In Section 2.6, we will show that it is equivalent to CAC for trees. Finally, the statement "for any 2-coloring of pairs, there exists an infinite set which is semi-hereditary for some color" is equivalent to RT^2_2 (see Corollary 2.6.13).

Restriction / Property	**Input**	**Output**
Transitivity	Equivalent to CAC	Weaker version of EM
Semi-ancestry	Equivalent to $\mathsf{RT}^1_{<\infty}$	Equivalent to RT^2_2
Semi-heredity	SHER, equivalent to CAC for trees	Equivalent to RT^2_2

Table 2.1: Summary of the equivalences for each forbidden pattern and restriction.

2.2 CAC for trees and its equivalences

In this section, we study some variations of CAC for trees and prove they are all equivalent. We also study TAC and show that TAC + $\mathsf{B}\Sigma^0_2$ is equivalent to CAC for trees. We start by defining these statements.

Statement 2.2.1 (CAC for c.e. (binary) trees)**.** Every infinite c.e. (binary) subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path or an infinite antichain.

In the context of reverse mathematics "being c.e." is a notion relative to the model considered. An object is c.e. when it can be approximated in a c.e. manner by objects from the model, as described in Section 1.2.7.

Definition 2.2.2. A node σ of a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a **split node** when there is $n_0, n_1 \in \mathbb{N}$ such that $\forall i < 2, \sigma \cdot n_i \in T$. In particular, if T is a binary tree, then σ is a split node when both $\sigma \cdot 0 \in T$ and $\sigma \cdot 1 \in T$. A tree $T \subseteq 2^{<\mathbb{N}}$ is **completely branching** when, for any of its node σ, if σ is not a leaf then it is a split node.

The following statement was introduced by Conidis [Con] (personal communication), motivated by the reverse mathematics of commutative noetherian rings.

Statement 2.2.3 (Tree AntiChain (TAC), [Con])**.** Any infinite c.e. subtree of $2^{<\mathbb{N}}$ which is completely branching, contains an infinite antichain.

Conidis [Con, Corollary 4.2] proved that TAC follows from ADS over RCA_0. He also constructed an instance of TAC whose solutions are all of hyperimmune degree [Con, Corollary 4.16], and used this result to prove that $\mathsf{WKL}_0 \nvdash \mathsf{TAC}$ [Con, Corollary 4.17]. We now proceed with the proof of the equivalence.

Theorem 2.2.4. *The following statements are equivalent over* RCA_0 *and computable reduction:*

(1) CAC for trees
(2) CAC for c.e. trees
(3) CAC for c.e. binary trees
(4) $\mathsf{TAC} + \mathsf{B}\Sigma^0_2$

Proof. (2) $\implies$ (1) and (2) $\implies$ (3) are immediate. (3) $\implies$ (4) is Proposition 2.2.5 and Proposition 2.2.6. (4) $\implies$ (2) is Proposition 2.2.7. (1) $\implies$ (2) is Proposition 2.2.8. □

We shall see in Section 2.3 that the use of $\mathsf{B}\Sigma^0_2$ is necessary for the above equivalence, as TAC does not imply $\mathsf{B}\Sigma^0_2$ over RCA_0.

Proposition 2.2.5. $\mathsf{RCA}_0 \vdash$ CAC for c.e. binary trees $\implies$ TAC *and* TAC $\leqslant_c$ CAC for c.e. binary trees.

Proof. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite completely branching c.e. tree. By the statement CAC for c.e. binary trees, either there is an infinite antichain, or an infinite path P. In the former case, we are done. In the latter case, using the fact that T is completely branching, the set $\{\sigma \cdot (1-i) : \sigma \cdot i \prec P\}$ is an infinite antichain of T. □

Proposition 2.2.6. $\mathsf{RCA}_0 \vdash$ CAC for c.e. binary trees $\implies \mathsf{RT}^1_{<\infty}$ *and* $\mathsf{RT}^1_{<\infty} \leqslant_c$ CAC for c.e. binary trees.

Proof. Let $f : \mathbb{N} \to k$ be a coloring, there are two possibilities. Either $\exists i < k, \exists^\infty x, f(x) = i$, in which case there is an infinite computable f-homogeneous set. Otherwise $\forall i < k, \forall^\infty x, f(x) \neq i$, in which case we define an infinite binary c.e. tree T, via a strictly increasing sequence of computable trees $(T_j)_{j\in\mathbb{N}}$ defined by $T_0 \coloneqq \{0^i : i < k\}$ and $T_{s+1} \coloneqq T_s \cup \{0^{f(s)} \cdot 1^{m+1}\}$, where m is the number of $x < s$ such that $f(x) = f(s)$.

Every antichain in T is of size at most k, thus, by CAC for c.e. binary trees, T must contain an infinite path, and so $\exists i < k, \exists^\infty x, f(x) = i$, which is a contradiction. □

Proposition 2.2.7. $\mathsf{RCA}_0 \vdash \mathsf{TAC} + \mathsf{B}\Sigma^0_2 \implies$ CAC for c.e. trees *and* CAC for c.e. trees $\leqslant_c$ TAC

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite c.e. tree. We can deal with two cases directly: if T has a node with infinitely many immediate children, then it contains a computable infinite antichain; and if T has finitely many split nodes, then it has finitely many paths $P_0, \dots, P_{k-1}$, which are all computable. Moreover one of them is infinite, as otherwise they would all be finite, i.e. $\forall i < k, \exists s, \forall n > s, n \notin P_i$, and thus their union would be finite since $\mathsf{B}\Pi^0_1$ (which is equivalent to $\mathsf{B}\Sigma^0_2$) yields $\exists b, \forall i < k, \exists s < b, \forall n > s, n \notin P_i$. But since $T = \bigcup_{i<k} P_i$, this would lead to a contradiction. For the remaining case, we define a **split triple** of T to be a triple $(\mu, n_0, n_1) \in T \times \mathbb{N} \times \mathbb{N}$ such that $\mu, \mu \cdot n_0, \mu \cdot n_1 \in T$. In particular, μ is a split node in T.

Idea. The general idea is to build greedily a completely branching c.e. tree S by looking for split triples in T, and mapping them to split nodes in S. This correspondence is witnessed by an injective function $f : S \to T$ that will be constructed alongside S. The main difficulty is that, since T is c.e., a split node ρ can be discovered after μ even though $\rho \prec \mu$, which means that we will not be able to ensure that S can be embedded in T. In particular, f will not be a tree morphism. However, the only property that needs to be ensured is that for every infinite antichain A of S, the set $f(A)$ will be an infinite antichain of T. To guarantee this, the function f needs to verify

$$\forall \sigma, \nu \in S, \sigma | \nu \implies f(\sigma) | f(\nu) \tag{$*$}$$

During the construction, at any step s, we are going to associate to each node $\sigma \in S$ a c.e. set $N^s_\sigma \subseteq T$, which might decrease in size over time ($N^0_\sigma \supseteq N^1_\sigma \supseteq \dots$), with the property that the elements of $\{N^s_\sigma : \sigma \in S\}$ are pairwise disjoint, and their union contains cofinitely many elements of T. The role of N^s_σ is to indicate that "if a split triple is found in N^s_σ, then the nodes in S, associated via f, must be above σ".

Construction. Initially, $N^0_\varepsilon := T$, $S := \{\varepsilon\}$ and $f(\varepsilon) := \varepsilon$. At step s, suppose we have defined a finite, completely branching binary tree $S \subseteq 2^{<\mathbb{N}}$, and for every $\sigma \in S$, a set $N^s_\sigma \subseteq T$ such that $\{N^s_\sigma : \sigma \in S\}$ forms a partition of T minus finitely many elements. Moreover, assume we have defined a mapping $f : S \to T$.

Search for a split triple (μ, n_0, n_1) in $\bigcup_{\sigma \in S} N^s_\sigma$. Let $\sigma \in S$ be such that $\mu \in N^s_\sigma$. Let τ be any leaf of S such that $\tau \succeq \sigma$ (for example pick the left-most successor of σ). Add $\tau \cdot 0$ and $\tau \cdot 1$ to S, and set $f(\tau \cdot i) = \mu \cdot n_i$ for each $i < 2$. Note that S is still completely branching.

Then, split N^s_σ into three disjoint subsets N^{s+1}_σ, $N^{s+1}_{\tau\cdot 0}$, $N^{s+1}_{\tau\cdot 1}$ as follows: for $i < 2$, $N^{s+1}_{\tau\cdot i} \coloneqq \{\rho \in N^s_\sigma : \rho \succcurlyeq \mu \cdot n_i\}$ and $N^{s+1}_\sigma \coloneqq \{\rho \in N^s_\tau : \forall i < 2, \rho | \mu \cdot n_i\}$. Note that these sets do not form a partition of N^s_σ as we missed the nodes in $\{\rho \in N^s_\sigma : \rho \preccurlyeq \mu\}$, fortunately there are only finitely many of them. Lastly, set $N^{s+1}_\nu \coloneqq N^s_\nu$ for every $\nu \in S - \{\sigma, \tau \cdot 0, \tau \cdot 1\}$.

Verification. First, let us prove that at any step s, $\bigcup_{\sigma\in S} N^s_\sigma$ contains infinitely many split triples, which ensures that the search always terminates. Note that $T - \bigcup_{\sigma\in S} N^s_\sigma$ is a Δ^0_2 subset of $\bigcup_{\sigma\in S}\{\rho : \rho \preccurlyeq \sigma\}$, hence exists by bounded Δ^0_2 comprehension, which follows from $\mathsf{B}\Sigma^0_2$. Moreover, by assumption, T is a finitely branching c.e. tree, which means that every $\rho \in T - \bigcup_{\sigma\in S} N^s_\sigma$ belongs to a bounded number of split triples of T. By $\mathsf{B}\Sigma^0_1$ (which follows from RCA_0), the number of split triples in T which involve a node from $T - \bigcup_{\sigma\in S} N^s_\sigma$ is bounded. Since by assumption, T contains infinitely many split triples, $\bigcup_{\sigma\in S} N^s_\sigma$ must contain infinitely many of them.

Second, we prove by induction on s that $\forall s \in \mathbb{N}, \forall \sigma \neq \nu \in S, N^s_\sigma | N^s_\nu$, i.e. $\forall \mu \in N^s_\sigma, \forall \rho \in N^s_\nu, \mu | \rho$. At step 0, the assertion is trivially verified. At step s, suppose we found the split triple (μ, n_0, n_1) in the set N^s_σ, and that $\forall i < 2, f(\tau \cdot i) = \mu \cdot n_i$ where $\tau \succcurlyeq \sigma$. Since μ was found in N^s_σ, the latter is split into $N^{s+1}_{\tau\cdot 0}$, $N^{s+1}_{\tau\cdot 1}$ and N^{s+1}_σ, the other sets remain identical. By construction, and because they are all subsets of N^s_σ, the assertion holds.

We now prove $(*)$, consider $\sigma, \nu \in S$ such that $\sigma | \nu$. WLOG suppose ν was added to S sooner than σ, more precisely $f(\nu)$ appeared (as child in a split triple) at step s in some set $N^s_{\_}$, so N^{s+1}_ν contains $f(\nu)$ by construction. Since σ was added to S after ν, there exists $\rho \in S$ such that $f(\sigma) \in N^{s+1}_\rho$. By contradiction $\rho \neq \sigma$ holds, as otherwise $f(\sigma) \in N^{s+1}_\tau$, and so σ would extend τ by construction of S. Thus by using the previous assertion, we deduce $f(\sigma) | f(\nu)$. ☐

Proposition 2.2.8. $\mathsf{RCA}_0 \vdash$ CAC for trees $\implies$ CAC for c.e. trees *and* CAC for c.e. trees $\leqslant_c$ CAC for trees.

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a c.e. tree. We define the computable tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$ by $\langle n_0, s_0\rangle \cdot \ldots \cdot \langle n_{k-1}, s_{k-1}\rangle \in S$ if and only if for all $j < k$, s_j is the smallest integer such that $n_0 \cdot \ldots \cdot n_j \in T[s_j]$, where $T[s_j]$ is the approximation of T at stage s_j.

By CAC for trees, there is an infinite chain (resp. antichain) in S, and by forgetting the second component of each string, we obtain an infinite chain (resp. antichain) in T. ☐

Before finishing this section, we introduce a set version of the principle TAC, which is more convenient to manipulate than TAC. Indeed, when working with TAC, the downward closure of the tree is not relevant, and we naturally end up taking an infinite computable subset of the tree rather than working with the c.e. tree. This motivates the following definitions.

Definition 2.2.9. A set $X \subseteq 2^{<\mathbb{N}}$ is **completely branching** if

$$\forall \sigma \in 2^{<\mathbb{N}}, (\sigma \cdot 0 \in X \iff \sigma \cdot 1 \in X)$$

Note that the above definition is compatible with the notion of completely branching tree.

Statement 2.2.10 (Set AntiChain (SAC))**.** Every infinite completely branching set $X \subseteq 2^{<\mathbb{N}}$ has an infinite antichain.

The set antichain theorem is equivalent to the tree antichain theorem, as the following lemma shows.

Lemma 2.2.11. $\mathsf{RCA}_0 \vdash \mathsf{SAC} \iff \mathsf{TAC}$.

Proof. SAC $\implies$ TAC. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite, completely branching c.e. tree. Let $S \subseteq T$ be an infinite computable, completely branching set. By SAC, there is an infinite antichain $A \subseteq X$. In particular, A is an antichain for T.

SAC $\impliedby$ TAC. Let $S \subseteq 2^{<\mathbb{N}}$ be an infinite computable, completely branching set. One can define an infinite computable tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ by letting $\sigma \in T$ iff for every $n < |\sigma|$, $\sigma(n)$ codes for a binary string $\tau_n \in S$, such that for every $n < |\sigma|-1$, $\tau_n \prec \tau_{n+1}$, and there is no string in S strictly between τ_n and τ_{n+1}. The tree T is such that every chain in T codes for a chain in S, and every antichain in T codes for an antichain in S. We can see T as an instance of CAC for trees. Moreover, since S is completely branching, then T has infinitely many split triples, so the proof of Proposition 2.2.7 applied to this instance T of CAC for trees does not use $\mathsf{B}\Sigma^0_2$. Thus there is either an infinite chain for T, or an infinite antichain for S. With the appropriate decoding, we obtain an infinite antichain for S. ☐

2.3 Probabilistic proofs of SAC

The restriction of CAC to trees yields a strictly weaker statement from the viewpoint of arithmetical bounds in the arithmetic hierarchy. Indeed, by Herrmann [Her01, Theorem 3.1], there is a computable partial order with no Δ^0_2 infinite chain or antichain, while by Binns et al. in [BKHL$^+$14, Theorem 6.2], every infinite computable tree must have either an infinite computable chain or an infinite Π^0_1 antichain. In this section, we go one step further in the study of the weakness of CAC for trees by proving that SAC admits probabilistic solutions. This result can also be obtained from the fact that TAC $\leqslant_c$ 2-RAN, which was proved by Conidis [Con, Corollary 4.8]. However, we provide a different and more direct proof here, relying on two technical lemmas.

Lemma 2.3.1 (RCA$_0$). *Let $S \subseteq 2^{<\mathbb{N}}$ be an infinite completely branching set. Then for every n, there exists an antichain of size n.*

Proof. By finite Ramsey's theorem for pairs and 2 colors (which holds in RCA$_0$), there exists some $p \in \mathbb{N}$ such that for every 2-coloring of $[p]^2$, there exists a homogeneous set of size n. Since S is infinite, there exists a subset $P \subseteq S$ of size p. By choice of p, there exists a subset $Q \subseteq P$ of size n such that Q is either a chain or an antichain. In the latter case, we are done. In the former case, since S is completely branching, the set $\{\widehat{\sigma} : \sigma \in Q\} \subseteq S$ is an antichain, where $\widehat{\sigma}$ is the string obtained from σ by flipping its last bit. □

Lemma 2.3.2 (RCA$_0$). *Let $S \subseteq 2^{<\mathbb{N}}$ be an infinite completely branching set. Then for every antichain $A \subseteq S$, for all but at most one $\sigma \in A$, the set $S_\sigma := \{\tau \in S : \sigma | \tau \text{ and } |\tau| > |\sigma|\}$ is infinite and completely branching.*

Proof. First, since $S \subseteq 2^{<\mathbb{N}}$ completely branching, then for every $\sigma \in 2^{<\mathbb{N}}$, the set S_σ is completely branching. Suppose for the sake of contradiction that there exists two strings $\sigma, \rho \in A$ such that S_σ and S_ρ are both finite. Then pick any $\tau \in S - (S_\sigma \cup S_\rho)$ with $|\tau| > \max(|\sigma|, |\rho|)$. It follows that $\sigma \prec \tau$ and $\rho \prec \tau$, and thus that σ and ρ are comparable, contradicting the fact that A is an antichain. □

Proposition 2.3.3. *The measure of the oracles computing a solution for a computable instance of* SAC *is 1.*

Remark 2.3.4. The proof of the above proposition is carried out purely as a computability-theoretic statement, hence we have access to as much induction as needed.

Proof. Let $S \subseteq 2^{<\mathbb{N}}$ be a computable and infinite completely branching set. We are going to build a decreasing sequence of infinite completely branching sets of strings $S_0 \supseteq S_1 \supseteq \dots$, with $S_0 \coloneqq S$, together with finite antichains $A_i \subseteq S_i$ (for $i \in \mathbb{N}$), in order to obtain an infinite antichain $A \coloneqq \{\sigma_i : i \in \mathbb{N}\}$ where $\sigma_i \in A_i$.

This construction will work with positive probability, and since the class of oracles computing a solution to the instance S is invariant under Turing equivalence, this implies that this class has measure 1. Indeed, by Kolmogorov's 0-1 law, every measurable Turing-invariant class has either measure 0 or 1.

First, let $S_0 \coloneqq S$. At step k, assume the sets $S_0 \supseteq S_1 \supseteq \dots \supseteq S_k$ and $A_0, \dots, A_{k-1}$ have been defined, as well as the finite antichain $\{\sigma_0, \dots, \sigma_{k-1}\}$, such that $\forall \tau \in S_k, \forall i < k, \sigma_i | \tau$.

Search computably for a finite antichain $A_k \subseteq S_k$ of size 2^{k+2}. If found, pick an element $\sigma_k \in A_k$ at random. Then define $S_{k+1} \coloneqq \{\tau \in S_k : \sigma_k | \tau \text{ and } |\tau| > |\sigma_k|\}$ for the next step.

If the procedure never stops, it yields an infinite antichain $A \coloneqq \{\sigma_i : i \in \mathbb{N}\}$ thanks to the definition of the sets $(S_i)_{i<k}$. Assuming that S_k is an infinite completely branching set, Lemma 2.3.1 ensures that A_k will be found.

However, if at any point, S_k is not an infinite completely branching set, then at some point t we will not be able to find a large enough A_t in it. If this happens, since S_{k+1} is completely determined by S_k and σ_k, it means that we have chosen some "bad" $\sigma_k \in A_k$. Luckily, by Lemma 2.3.2, there is at most one element of this kind in A_k. Thus, if we pick σ_k at random in A_k, we have at most $\frac{1}{|A_k|} = \frac{1}{2^{k+2}}$ chances for this case to happen. So the overall probability that this procedure fails is less than $\sum_{k \geqslant 0} \frac{1}{2^{k+2}} = \frac{1}{2}$. Hence we found an antichain with positive probability. □

Very few theorems studied in reverse mathematics admit a probabilistic proof. Proposition 2.3.3 provides a powerful method for separating the statement CAC for trees from many theorems in reverse mathematics. In what follows, AMT stands for the Atomic Model Theorem, studied by Hirschfeldt, Shore, and Slaman [HSS09], COH is the cohesiveness principle, defined by Cholak, Jockusch, and Slaman [CJS01, Statement 7.7], and RWKL is the Ramsey-type Weak König's lemma, defined by Flood [Flo12, Statement 2] under the name RKL.

Corollary 2.3.5. *Over* RCA_0*,* CAC for trees *implies none of* AMT*,* COH *and* RWKL*.*

Proof. These three statements have a computable instance such that the measure of the oracles computing a solution is 0, see Astor et al. [ABD+]. ☐

The argument in the proof of Proposition 2.3.3 can be formalized over RCA_0 to yield the following result. Conidis [Con, Corollary 4.8] independantly proved a slightly weaker version, namely $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 \vdash 2\text{-}\mathsf{RAN} \implies \mathsf{SAC}$.

Statement 2.3.6 (2-RAN)**.** For every sequence of uniformly Π^0_2 binary trees $T_0, T_1, \dots$ such that, for every n, $\mu([T_n]) > 1 - 2^{-n}$, there is some n and some set X such that $X \in [T_n]$.

Proposition 2.3.7. $\mathsf{RCA}_0 \vdash 2\text{-}\mathsf{RAN} \implies \mathsf{SAC}$.

Proof. For every n, consider the construction of Proposition 2.3.3, where the antichain A_k is of size 2^{n+k+1} instead of 2^{k+2}. For each k, let $\sigma_k \in A_k$ be the unique "bad" choice (if it exists), that is, which makes the set S_{k+1} finite, and let τ_k be the string of length $n + k + 1$ corresponding to the binary representation of the rank of σ_k in A_k for some fixed order on binary strings. Then one can compute σ_k from τ_k and the finite set A_k. Note that τ_k is undefined when σ_k does not exist.

Consider the Σ^0_2 class $\mathcal{U}_n := \{X \in 2^{\mathbb{N}} : \exists k, \tau_k \prec X\} = \bigcup_k [\tau_k]$. It verifies

$$\mu(\mathcal{U}_n) \leqslant \sum_{\substack{k \geqslant 0 \\ \sigma_k \text{ exists}}} \mu\big([\tau_k]\big) \leqslant \sum_{k \geqslant 0} \frac{1}{2^{n+k+1}} = 2^{-n}$$

Let T_n be a Π^0_2 tree such that $[T_n] = 2^{\mathbb{N}} - \mathcal{U}_n$. We can now consider the sequence of trees $(T_n)_{n \in \mathbb{N}}$. By 2-RAN, there is some n and some $X \in [T_n]$. For any instance of SAC, find a solution by running the construction given in Proposition 2.3.3 with the help of X to avoid the potential "bad" choice in each A_k. ☐

Corollary 2.3.8. *Over* RCA_0, SAC *(and therefore* TAC*) implies none of* $\mathsf{B}\Sigma^0_2$ *and* CAC for trees.

Proof. Slaman [Sla11] proved that 2-RAN does not imply $\mathsf{B}\Sigma^0_2$ over RCA_0. The corollary follows from $\mathsf{RCA}_0 \vdash \mathsf{SAC} \implies \mathsf{TAC}$ (Lemma 2.2.11) and $\mathsf{RCA}_0 \vdash \mathsf{TAC} + \mathsf{B}\Sigma^0_2 \iff$ CAC for trees (Theorem 2.2.4). ☐

We are now going to refine Proposition 2.3.3 by proving that some variant of DNC is sufficient to compute a solution of SAC.

Definition 2.3.9 (Diagonally non-computable function)**.** A function $f : \mathbb{N} \to \mathbb{N}$ is **diagonally non-computable relative to** X (or $\mathsf{DNC}(X)$) if for every e, $f(e) \neq \Phi_e^X(e)$. Whenever f is dominated by a function $h : \mathbb{N} \to \mathbb{N}$, then we say that f is $\mathsf{DNC}_h(X)$. A Turing degree is $\mathsf{DNC}_h(X)$ if it contains a $\mathsf{DNC}_h(X)$ function.

The following lemma gives a much more convenient way to work with $\mathsf{DNC}_h(X)$ functions.

Lemma 2.3.10 (Folklore)**.** *Let A, X be subsets of $\mathbb{N}$. The following are equivalent:*

(1) A is of degree $\mathsf{DNC}_h(X)$ for some computable (primitive recursive) function $h : \mathbb{N} \to \mathbb{N}$.

(2) A computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\forall e, n, |W_e^X| \leqslant n \implies g(e,n) \notin W_e^X$$

and which is dominated by a computable function $b : \mathbb{N}^2 \to \mathbb{N}$, i.e.

$$\forall e, n, g(e,n) < b(e,n)$$

Proof. (2) $\implies$ (1). Let $i : \mathbb{N} \to \mathbb{N}$ be a computable (primitive recursive) function such that for any $e \in \mathbb{N}$ and $B \subseteq \mathbb{N}$ we have $\Phi_{i(e)}^B(x) \downarrow \iff x = \Phi_e^B(e)$. Thus

$$W_{i(e)}^B = \begin{cases} \left\{\Phi_e^B(e)\right\} & \text{if } e \in B' \\ \varnothing & \text{otherwise} \end{cases}$$

From there, define the A-computable function $f : \mathbb{N} \to \mathbb{N}$ by $f : e \mapsto g(i(e), 1)$. It is $\mathsf{DNC}(X)$ because $g(i(e),1) \notin W_{i(e)}^X$ since $|W_{i(e)}^X| \leqslant 1$. Moreover, f is dominated by the computable function $e \mapsto b(i(e), 1)$, because b computably dominates g.

(1) $\implies$ (2). Let f be a $\mathsf{DNC}_h(X)$ function computed by A. Given the pair e, n, we describe the process that defines $g(e,n)$.

Construction. For each $i < n$, we compute the code $u(e,i)$ of the X-computable function which, on any input, looks for the i^{th} element of W_e^X. If it finds such an element, then it interprets it as an n-tuple $\langle k_0, \dots, k_{n-1}\rangle$ and returns the value k_i. If it never finds such an element, then the function diverges. Finally we define $g : e, n \mapsto \langle f(u(e,0)), \dots, f(u(e,n-1))\rangle$

Verification. First, since f is dominated by h, and since the function $\langle -, \dots, - \rangle$ computing an n-tuple is increasing on each variable, we can dominate g with the computable function

$$b : e, n \mapsto \left\langle h(u(e,0)), \dots, h(u(e,n-1)) \right\rangle$$

Now, by contradiction, suppose g does not satisfy (2), i.e. suppose there exists e, n such that $|W_e^X| \leqslant n$ but $g(e,n) \in W_e^X$. Because W_e^X has fewer than n elements, we can suppose $g(e,n)$ is the i^{th} one for a some $i < n$. Thus the function $\Phi^X_{u(e,i)}$ is constantly equal to k_i where $g(e,n) = \langle k_0, \dots, k_{n-1} \rangle$, in particular $\Phi^X_{u(e,i)}(u(e,i)) = k_i$. But we also have

$$g(e,n) = \left\langle f(u(e,0)), \dots, f(u(e,n-1)) \right\rangle$$

implying $f(u(e,i)) = k_i = \Phi^X_{u(e,i)}(u(e,i))$, which is impossible as f is supposed to be $\mathsf{DNC}_h(X)$. □

Before continuing, we also need a classic result of computability theory.

Lemma 2.3.11 (Kleene Fixed-Point Theorem)**.** *For any total computable function $f : \mathbb{N} \to \mathbb{N}$, there is an index e such that*

$$\Phi_{f(e)} = \Phi_e$$

We are now ready to prove the following proposition. Conidis [Con] independently proved the same statement for TAC with a similar construction. Note that by the equivalence of $\mathsf{TAC} + \mathsf{B}\Sigma^0_2$ with CAC for trees, Conidis' result implies Proposition 2.3.12.

Proposition 2.3.12. *Let $S \subseteq \mathbb{N}^{<\mathbb{N}}$ be an instance of* SAC*. For any computable function h, we have that every set X of degree $\mathsf{DNC}_h(\varnothing')$ computes a solution of S.*

Remark 2.3.13. Once again, as in the case of Proposition 2.3.3, the proof here is carried purely as a computability-theoretic statement, we have access to as much induction as we need.

Proof. First, since X is of degree DNC_h for a computable function h, by Lemma 2.3.10, it computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that $\forall e, n, |W_e^{\varnothing'}| \leqslant n \implies g(e,n) \notin W_e^{\varnothing'}$ and which is dominated by a computable function $b : \mathbb{N}^2 \to \mathbb{N}$.

The idea of this proof is the same as in Proposition 2.3.3, but this time we are going to use g to avoid selecting the potential "bad" element in each finite antichain, i.e. the element which is incompatible with only finitely many strings. For any finite set $A \subseteq S$, let $\psi_A : \mathbb{N} \to S$ be a bijection such that $\psi_A(\llbracket 0, |A| \llbracket) = A$.

The procedure is the following. Initially, $S_0 \coloneqq S$. At step k, assume $S_k \subseteq S$ has been defined. To find the desired antichain A_k we use Kleene Fixed-Point Theorem to find an index e_k such that $\Phi_{e_k}^{\varnothing'}(n)$ is the procedure that halts if it finds an antichain $A \subseteq S_k$ whose size is greater than $b(e_k, 1)$ and $\psi_A(n) \in A$, and finds (using $\varnothing'$) an integer m such that $\forall \ell > m, \psi_A(\ell) \succ \psi_A(n)$.

Define $A_k \coloneqq A$. By choice of A and e_k,

$$W_{e_k}^{\varnothing'} = \begin{cases} \{\psi_A^{-1}(\rho)\} & \text{if } A_k \text{ has a bad element } \rho \\ \varnothing & \text{otherwise} \end{cases}$$

Finally we can define $\sigma_k \coloneqq \psi_A(g(e_k, 1))$. Indeed since $|W_{e_k}^{\varnothing'}| \leqslant 1$ by construction, $g(e_k, 1) \notin W_{e_k}^{\varnothing'}$. Moreover $\sigma_k \in A_k$, because $g(e_k, 1) < b(e_k, 1) < |A_k|$. This implies that σ_k is not a bad element of A_k, in other words the set $S_{k+1} \coloneqq \{\tau \in S_k : \tau | \sigma_k \text{ and } |\tau| > |\sigma_k|\}$ is infinite. □

2.4 Relation between CAC for trees and ADS + EM

Ramsey's theorem for pairs admits a famous decomposition into the Ascending Descending Sequence theorem (ADS) and the Erdős-Moser theorem (EM) over RCA_0. As mentioned in the introduction, both statements are strictly weaker than RT^2_2. These statements are generally thought of as decomposing Ramsey's theorem for pairs into its disjunctiveness part with ADS, and its compactness part with EM. Indeed, the standard proof of ADS is disjunctive and does not involve any notion of compactness, while the proof of EM is non-disjunctive and implies RWKL, which is the compactness part of RT^2_2.

ADS and EM are relatively disjoint, in that they are only known to have the hyperimmunity principle as a common consequence, which is a particularly weak principle. In this section however, we show that CAC for trees follows from both ADS and EM over RCA_0. We shall see in Section 2.5 that CAC for trees implies the hyperimmunity principle.

The following proposition was proved by Dorais (personal communication) for SHER, we here give an adaptation of this proof for CAC for trees. Besides, an alternative proof can be obtained by combining Conidis' result that ADS implies TAC over RCA_0 [Con, Corollary 4.2] and Proposition 2.2.7, since ADS implies $\mathsf{B}\Sigma^0_2$.

Proposition 2.4.1. $\mathsf{RCA}_0 \vdash \mathsf{ADS} \implies$ CAC for trees *and* CAC for trees $\leqslant_c$ ADS

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite tree. We denote by $<_0$ the lexicographic order over T, i.e. the total order defined on T by $\sigma <_0 \tau \iff \sigma \prec \tau \vee (\sigma|\tau \wedge \sigma(d) <_{\mathbb{N}} \tau(d))$ where $d \coloneqq \min\{k \in \mathbb{N} : \sigma(k) \neq \tau(k)\}$. By ADS, there is an infinite ascending or descending sequence $(\sigma_i)_{i\in\mathbb{N}}$ for $(T, <_0)$.

If it is descending, there are two possibilities. Either $\forall^\infty i, \sigma_i \not| \sigma_{i+1}$, which means we eventually have an infinite $\prec$-decreasing sequence of strings, which is impossible. Or $\exists^\infty i, \sigma_i|\sigma_{i+1}$, in which case we have a sequence $(\ell_k)_{k\in\mathbb{N}}$ of indexes such that $\forall k, \sigma_{\ell_k}|\sigma_{\ell_k+1}$, and we designate by $(h_k)_{k\in\mathbb{N}}$ the sequence $(\sigma_{\ell_k+1})_{k\in\mathbb{N}}$, and we show that it is an antichain of T.

To do so, it suffices to prove by induction on m that $\forall m > 0, \forall k, h_k \not\succ h_{k+m}$. When $m = 1$, due to how $(\sigma_i)_{i\in\mathbb{N}}$ is structured, we have $\forall k, h_k \not\succcurlyeq \sigma_{\ell_{k+1}}$ and by definition $\sigma_{\ell_{k+1}}|h_{k+1}$, thus $h_k \not\succ h_{k+1}$. We now consider h_k and $h_{k+(m+1)}$. By induction hypothesis, $h_k \not\succ h_{k+m}$ and $h_{k+m} \not\succ h_{(k+m)+1}$. Moreover since $h_k >_0 h_{k+m} >_0 h_{k+(m+1)}$, we know there are minima d and e such that $h_k(d) > h_{k+m}(d)$ and $h_{k+m}(e) > h_{k+(m+1)}(e)$. Now $e < d$ implies $h_{k+(m+1)}(e) < h_{k+m}(e) = h_k(e)$, $e > d$ implies $h_{k+(m+1)}(d) = h_{k+m}(d) < h_k(d)$, and $e = d$ implies $h_{k+(m+1)}(e) < h_{k+m}(e) < h_k(e)$; in any case $h_k \not\succ h_{k+(m+1)}$.

Now if the sequence $(\sigma_i)_{i\in\mathbb{N}}$ is ascending, we again distinguish two possibilities. Either $\forall^\infty i, \sigma_i \not| \sigma_{i+1}$, which means we eventually obtain an infinite path of the tree. Or $\exists^\infty i, \sigma_i|\sigma_{i+1}$, in which case we work in the same fashion as in the descending case: designate by $(h_k)_{k\in\mathbb{N}}$ the sequence $(\sigma_{\ell_k})_{k\in\mathbb{N}}$ of all such σ_i, and show by induction on m that $\forall m > 0, \forall k, h_k \not\prec h_{k+m}$. □

Remark 2.4.2. Note that the above proof also works if we define $<_0$ to be the Kleene–Brouwer order on T, i.e. the total order defined on T by $\sigma <_0 \tau \iff \sigma \succ \tau \vee (\sigma|\tau \wedge \sigma(d) <_{\mathbb{N}} \tau(d))$ where $d \coloneqq \min\{k \in \mathbb{N} : \sigma(k) \neq \tau(k)\}$.

Corollary 2.4.3. TAC *does not imply 2*-DNC *nor 2*-RAN

Proof. By Proposition 2.4.1 we have that ADS implies TAC, but ADS does not imply 2-DNC ([HS07, Corollary 2.28]), hence neither does TAC. Moreover, since 2-RAN implies 2-DNC ([BPS17, Theorem 2.8]), we have that TAC does not imply RAN either. □

Proposition 2.4.4. $\mathsf{RCA}_0 \vdash \mathsf{EM} \implies$ CAC for trees *and* CAC for trees $\leqslant_c$ EM

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite tree. We first define a T-computable bijection $\psi : \mathbb{N} \to T$. To do so, let $\varphi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be the bijection $x_0 \cdot \ldots \cdot x_{n-1} \mapsto \left(\Pi_{i<n} p_i^{x_i+1}\right) - 1$ where p_i is the i^{th} prime number. The elements of the sequence $(\varphi^{-1}(n))_{n\in\mathbb{N}}$ that are in T form a subsequence denoted $(s_n)_{n\in\mathbb{N}}$, and the function $\psi : \mathbb{N} \to T$ is defined by $n \mapsto s_n$. Note that the range of ψ is T. Moreover, if $\sigma \prec \tau \in T$, then $\varphi(\sigma) < \varphi(\tau)$, hence $\psi^{-1}(\sigma) < \psi^{-1}(\tau)$.

Let $f : [\mathbb{N}]^2 \to 2$ be the coloring defined by $f(\{x, y\}) = 1$ iff $x <_{\mathbb{N}} y$ and $\psi(x) \prec \psi(y)$ coincide. By EM, there is an infinite transitive set $S \subseteq \mathbb{N}$, i.e. $\forall i < 2, \forall x<y<z \in S, f(x, y) = f(y, z) = i \implies f(x, z) = i$

Note that if there are $x<y \in S$ such that $f(x, y) = 0$, then $\forall z>y \in S, f(x, z) = 0$. Indeed given $x<y<z \in S$ such that $f(x, y) = 0$, either $f(y, z) = 0$, and so by transitivity we have $f(x, z) = 0$; or $f(y, z) = 1$, but in that case $f(x, z) \neq 1$ because it is impossible to simultaneously have $\psi(y) \prec \psi(z)$, $\psi(x) \prec \psi(z)$ and $\psi(x)|\psi(y)$.

Now two cases are possible. Either $\exists^\infty j \in \mathbb{N}, f(s_j, s_{j+1}) = 0$, so consider the infinite set A made of all such s_j. Thanks to the previous property, A is f-homogeneous for the color 0, and so $\psi(A)$ is an infinite antichain. Or $\forall^\infty j \in \mathbb{N}, f(s_j, s_{j+1}) = 1$, so there is a large enough $k \in \mathbb{N}$ such that $\psi(s_k) \prec \psi(s_{k+1}) \prec \ldots$, i.e. we found an infinite path. □

Corollary 2.4.5. *The implication between* ADS*,* EM *and* CAC for trees *are strict.*

Proof. CAC for trees does not imply ADS nor EM because, on one hand, by Corollary 2.3.5, it does not imply AMT nor RWKL, and on the other hand, ADS implies AMT ([HSS09, Theorem 4.1]) and EM implies RWKL ([BPS17, Theorem 2.11] and [FT16, Theorem 5.2]). □

2.5 TAC, lowness and hyperimmunity

Binns et al. in [BKHL+14] and Conidis [Con] respectively studied the reverse mathematics of CAC for trees and TAC. Since CAC for trees is computably equivalent to TAC and this equivalence also holds in reverse mathematics over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, the analysis of CAC for trees and TAC is very similar. For example, Binns et al. [BKHL+14, Theorem 6.4] proved that for any fixed low set L, there is a computable instance of CAC for trees with no L-computable solution, while Conidis

[Con] proved the existence of a computable instance of TAC whose solutions are all of hyperimmune degree. In this section, we prove a general statement regarding TAC (Theorem 2.5.1) and show that it encompasses both results.

Theorem 2.5.1. *Let $(A_n)_{n\in\mathbb{N}}$ be a uniformly Δ^0_2 sequence of infinite Δ^0_2 sets. There is a computable instance of* TAC *such that no A_n is a solution.*

Proof. First, for any n, let e_n be the index of A_n, i.e. $\Phi^{\varnothing'}_{e_n} = A_n$. We also write $A_n[s] \coloneqq \Phi^{\varnothing'[s]}_{e_n}[s]$.

Idea. We are going to construct a tree $T \subseteq 2^{<\mathbb{N}}$, such that for each $n \in \mathbb{N}$, there is $\sigma_n \in A_n$ verifying $\sigma_n \notin T$ or $\sigma_n \in T \wedge \forall^\infty \tau \in T, \sigma_n \prec \tau$. These requirements are respectively denoted $\mathcal{R}_n$ and $\mathcal{S}_n$, and A_n cannot be an infinite antichain of T if one of them is met.

The sequence (σ_n) is constructed via a movable marker procedure, with steps s and sub-steps $e < s$. At each step s we are going to manipulate an approximation σ^s_n of σ_n, and variables $\widehat{\sigma}^s_n$ that will help us keep track of which requirement is satisfied by σ^s_n.

Construction. At the beginning of each step s, let T_s be the approximation of the tree T defined by $T_s \coloneqq T_{s-1} \cup \{\tau_s \cdot 0, \tau_s \cdot 1\}$ where τ_s is the leftmost (for example) leaf of T_{s-1} such that $\tau_s \succcurlyeq \widehat{\sigma}^s_{s-1}$. For $s = 0$, we let $T_0 \coloneqq \{\varepsilon\}$.

At step s, sub-step e, let σ^s_e be the string whose code is the smallest in the uniformly computable set $\{\tau \in A_e[s]{\upharpoonright}_s : (\tau \in T_s \wedge \tau \succcurlyeq \widehat{\sigma}^s_{e-1}) \vee \tau \notin T_s\}$ with $\widehat{\sigma}^s_{-1} \coloneqq \varepsilon$ and σ^s_e is undefined when the set is empty.

Besides, define $\widehat{\sigma}^s_e \coloneqq \begin{cases} \sigma^s_e & \text{if } \sigma^s_e \in T_s \text{ (and therefore } \sigma^s_e \succcurlyeq \widehat{\sigma}^s_{e-1}) \\ \widehat{\sigma}^s_{e-1} & \text{otherwise} \end{cases}$

Verification. By induction on e, we prove that $\sigma_e \coloneqq \lim_s \sigma^s_e$ exists and is an element of A_e, also we prove $\widehat{\sigma}_e \coloneqq \lim_s \widehat{\sigma}^s_e$ exists, and σ_e satisfies $\mathcal{R}_e$ or $\mathcal{S}_e$.

Suppose we reached a step r such that for all $e' < e$ the values of $\sigma^r_{e'}$ and $\widehat{\sigma}^r_{e'}$ have stabilized. And thus, for any step $s > r$, as $\tau_s \succcurlyeq \widehat{\sigma}^s_{s-1} \succcurlyeq \widehat{\sigma}^s_{e-1} = \widehat{\sigma}_{e-1}$, the tree will always be extended with nodes above $\widehat{\sigma}_{e-1}$, implying only a finite part of the tree is not above $\widehat{\sigma}_{e-1}$.

Now suppose k is the smallest code of a string τ such that $(\tau \in T \wedge \tau \succcurlyeq \widehat{\sigma}_{e-1}) \vee \tau \notin T$. Such a string exists because A_e is infinite, whereas the set of strings in T that are below $\widehat{\sigma}_{e-1}$ is not. If $\tau \in T$, then $\exists x, \forall y \geqslant x, \tau \in T_y$, otherwise define $x \coloneqq 0$. Since A_e is Δ^0_2, there exists $s \geqslant \max\{k+1, r, x\}$ such that $A_e[s]{\upharpoonright}_{k+1}$ has stabilized i.e. $\forall t > s, A_e[t]{\upharpoonright}_{k+1} = A_e[s]{\upharpoonright}_{k+1}$. Thus $\sigma^s_e = \tau$ because $\tau \in A_e{\upharpoonright}_{k+1} = A_e[s]{\upharpoonright}_{k+1} \subseteq A_e[s]{\upharpoonright}_s$. This ensures that for any $t > s$, $\sigma^t_e = \tau$, i.e. $\sigma_e = \tau$.

Finally, we distinguish two cases. Either $\sigma_e \in T$ and so $\exists t, \sigma_e^t \in T_t$, thus $\forall u > t, \widehat{\sigma}_e^u = \sigma_e^u$. So $\mathcal{S}_e$ is satisfied, as cofinitely many nodes of T will be above $\widehat{\sigma}_e = \sigma_e$. Or $\sigma_e \notin T$, in which case, either $\forall t, \sigma_e^t \notin T_t$, implying $\forall t, \widehat{\sigma}_e^t \coloneqq \widehat{\sigma}_{e-1}^t$ and thus $\mathcal{R}_e$ is satisfied. □

We now show how Theorem 2.5.1 relates to the result of Binns et al. in [BKHL+14, Theorem 6.4], that is, the existence, for any fixed low set L, of a computable instance of CAC for trees with no L-computable solution.

Lemma 2.5.2. *For any low set P, the sequence of infinite P-computable sets is uniformly Δ_2^0.*

Proof. Since $P' \leqslant_T \varnothing'$ we can $\varnothing'$-compute the function

$$f(e,x) = \begin{cases} \Phi_e^P(x) & \text{when } \forall y \leqslant x, \Phi_e^P(y)\downarrow \text{ and } \exists y > x, \Phi_e^P(y)\downarrow = 1 \\ 1 & \text{otherwise} \end{cases}$$

Now let $A_e \coloneqq \{f(e,x) : x \in \mathbb{N}\}$. If Φ_e^P is total and infinite then A_e is equal to it, so it is P-computable. Otherwise A_e is cofinite, and in particular it is infinite and P-computable. □

We are now ready to state the result of Binns et al. in [BKHL+14, Theorem 6.4], but for TAC.

Corollary 2.5.3. *For any low set P, there exists a computable instance of* TAC *with no P-computable solution.*

Proof. Given P, we can use Lemma 2.5.2 to obtain a uniform sequence, on which we apply Theorem 2.5.1. □

The previous corollary is very useful to show that $\mathsf{WKL}_0 \nvdash \mathsf{TAC}$ since there exists a model of WKL_0 below a low set. It also proves that $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2^0 \nvdash \mathsf{TAC}$, as COMP is a model of $\mathsf{B}\Sigma_2^0$ but not TAC. The following corollary will be useful to prove that the result of Binns et al. in [BKHL+14, Theorem 6.4] implies the result of Conidis.

Corollary 2.5.4. *There exist a* PA *degree P and an instance of* TAC *with no P-computable solution.*

Proof. It follows from the existence of a low PA degree by the low basis theorem, see [JS72a, Corollary 2.2.]. □

The next proposition has two purposes. First, it will be used to show the existence of a computable instance of TAC whose solutions are all of hyperimmune degree (see Theorem 2.5.6). Second, it shows that, for any such instance, one can choose two specific functionals to witness this hyperimmunity, without loss of generality (see Corollary 2.5.8).

Proposition 2.5.5. *Let T be an instance of* TAC*. For any set P of* PA *degree, if T has no P-computable solution, then for any solution $(\sigma_n)_{n\in\mathbb{N}}$, the function $t_{T,(\sigma_n)_{n\in\mathbb{N}}} : n \mapsto \min\{t : \sigma_n \in T[t]\}$ or $\ell_{(\sigma_n)_{n\in\mathbb{N}}} : n \mapsto |\sigma_n|$ is hyperimmune.*

Proof. By contraposition, suppose there exists a solution $(\sigma_n)_{n\in\mathbb{N}}$ such that $t_{T,(\sigma_n)_{n\in\mathbb{N}}}$ and $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ are computably dominated by t and ℓ respectively. Then the set

$$\left\{(\tau_n)_{n\in\mathbb{N}} \ : \ \begin{array}{l} (\tau_n)_{n\in\mathbb{N}} \text{ is an infinite antichain of } T \\ t_{T,(\tau_n)_{n\in\mathbb{N}}} \leqslant t \text{ and } \ell_{(\tau_n)_{n\in\mathbb{N}}} \leqslant \ell \end{array}\right\}$$

is a non-empty Π^0_1 class. It is non-empty because $(\sigma_n)_{n\in\mathbb{N}}$ belongs to it, and to show it is a Π^0_1 class, it can be written as

$$\left\{(\tau_n)_{n\in\mathbb{N}} \ : \ \forall n, \ \begin{array}{l} \forall m < n, \tau_n | \tau_m \\ \tau_n \in T[t(n)] \\ |\tau_n| \leqslant \ell(n) \end{array}\right\}$$

Thanks to ℓ, the number of elements at each level n of the tree associated to this class is computably bounded by $2^{\ell(n)}$, thus it can be coded by a Π^0_1 class of $2^{\mathbb{N}}$. Finally, since P is of PA degree, it computes an element of any Π^0_1 class of the Cantor space, hence the result. □

Combining Corollary 2.5.4 and Proposition 2.5.5, we obtain the following theorem from Conidis [Con].

Theorem 2.5.6 (Conidis [Con])**.** *There is a computable instance of* TAC *such that each solution is of hyperimmune degree.*

Proof. Let P be of low PA degree. By using Corollary 2.5.3 we get a computable instance T of TAC with no P-computable solution. Thus, by using Proposition 2.5.5 we deduce that, for any solution (σ_n), its function $t_{T,(\sigma_n)_{n\in\mathbb{N}}}$ or $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ is hyperimmune. And $(\sigma_n)_{n\in\mathbb{N}}$ computes both, since T is computable ; meaning it is of hyperimmune degree. □

Corollary 2.5.7. $\mathsf{RCA}_0 \vdash \mathsf{TAC} \implies \mathsf{HYP}$

In his direct proof of Theorem 2.5.6, Conidis [Con] constructed computable instance of TAC and two functionals Φ, Ψ such that for every solution H, either Φ^H or Ψ^H is hyperimmune. Interestingly, Proposition 2.5.5 can be used to show that Φ and Ψ can be chosen to be $t_{T,-}$ and ℓ_-, without loss of generality.

Corollary 2.5.8. *For any instance T of* TAC *whose solutions are all of hyperimmune degree, at least one of the function $t_{T,-}$ or ℓ_- is a witness.*

Proof. Let T be an instance of TAC whose solutions are all of hyperimmune degree, and let $(\sigma_n)_{n\in\mathbb{N}}$ be such a solution. By contradiction, if we suppose $t_{T,(\sigma_n)_{n\in\mathbb{N}}}$ and $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ are both computably dominated, then by Proposition 2.5.5, T has a P-computable solution. If we choose P to be computably dominated, then it cannot compute a solution of hyperimmune degree, hence a contradiction. □

Note that for every (computable or not) instance of TAC, there is a solution $(\sigma_n)_{n\in\mathbb{N}}$ such that $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ is dominated by the identity function, by picking any path, and building an antichain along it.

2.6 Equivalence between CAC for trees and SHER

We have seen in Section 2.4 that CAC for trees follows from both ADS and EM over RCA_0. The proof of CAC for trees from ADS used only one specific property of the partial order $(T, \prec)$, that we shall refer to as **semi-heredity**. Dorais and al. [DDH+16] introduced the principle SHER, which is the restriction of Ramsey's theorem for pairs to semi-hereditary colorings. In this section, we show that the seemingly artificial principle SHER turns out to be equivalent to the rather natural principle CAC for trees. This equivalence can be seen as one more step towards the robustness of CAC for trees.

Definition 2.6.1. A coloring $f : [\mathbb{N}]^2 \to 2$ is **semi-hereditary** for the color $i < 2$ if

$$\forall x<y<z, f(x,z) = f(y,z) = i \implies f(x,y) = i$$

The name "semi-heredity" comes from the contraposition of the previous definition $\forall x<y<z, f(x,y) = 1-i \implies f(x,z) = 1-i \vee f(y,z) = 1-i$

Definition 2.6.2 (SHER, [DDH+16])**.** For any semi-hereditary coloring f, there exists an infinite f-homogeneous set.

The first proposition consists essentially of noticing that, given a set of strings $T \subseteq \mathbb{N}^{<\mathbb{N}}$, the partial order $(T, \prec)$ behaves like a semi-hereditary coloring. The whole technicality of the proposition comes from the definition of an injection $\psi : \mathbb{N} \to T$ with some desired properties.

Proposition 2.6.3. $\mathsf{RCA}_0 \vdash \mathsf{SHER} \implies$ CAC for c.e. trees *and* CAC for c.e. trees $\leqslant_c$ SHER

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite c.e. tree. First, let $\varphi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ the bijection $x_0 \cdot \ldots \cdot x_{n-1} \mapsto p_0^{x_0} \times \ldots \times p_{n-1}^{x_{n-1}} - 1$ where p_k is the k^{th} prime number. Define $\psi : \mathbb{N} \to T$ by letting $\psi(n)$ be the least $\sigma \in T$ (in order of apparition) such that $\phi(\sigma)$ is bigger than $\phi(\psi(0)), \phi(\psi(1)), \ldots, \phi(\psi(n-1))$. Note that, by construction, the range of ψ is infinite and computable. Moreover, if $\sigma \prec \tau$, then $\varphi(\sigma) < \varphi(\tau)$, hence $\psi^{-1}(\sigma) < \psi^{-1}(\tau)$. Also, note that the range of ψ is not necessarily a tree.

Let $f : [\mathbb{N}]^2 \to 2$ be the coloring defined by $f(\{x, y\}) = 1$ iff $x <_{\mathbb{N}} y$ and $\psi(x) \prec \psi(y)$ coincide. Let us show that f is semi-hereditary for the color 1. Suppose we have $x < y < z$ and that $f(x, z) = f(y, z) = 1$, i.e. letting $\sigma := \psi(x), \tau := \psi(y), \rho := \psi(z)$ then we have $\sigma \prec \rho$ and $\tau \prec \rho$, thus either $\sigma \prec \tau$ or $\tau \prec \sigma$. But since $x < y$, i.e. $\psi^{-1}(\sigma) < \psi^{-1}(\tau)$, only $\sigma \prec \tau$ can hold due to the above note, meaning $f(x, y) = 1$.

By SHER applied to f, there is an infinite f-homogeneous set H. If it is homogeneous for the color 0, then the set $\psi(H)$ corresponds to an infinite antichain of T. Likewise, if it is homogeneous for the color 1, then the set $\psi(H)$ is an infinite path of T. □

We now prove the converse of the previous proposition.

Definition 2.6.4. Given a coloring $f : [\mathbb{N}]^2 \to k$, a set $A := \{a_0 < a_1 < \ldots\} \subseteq \mathbb{N}$ is **weakly-homogeneous** for the color $i < k$ if $\forall j, f(a_j, a_{j+1}) = i$

Before proving Proposition 2.6.6, we need a technical lemma.

Lemma 2.6.5 (RCA_0)**.** *Let $f : [\mathbb{N}]^2 \to 2$ be a semi-hereditary coloring for the color $i < 2$. For every infinite set $A := \{a_0 < a_1 < \ldots\}$ which is weakly-homogeneous for the color i, there is an infinite f-homogeneous subset $B \subseteq A$.*

Proof (Dorais). We first show that any a_j falls in one of these two categories:

1. $\forall k > j, f(a_j, a_k) = i$
2. $\exists \ell > j, \big(\forall k \in \rrbracket j, \ell \llbracket, f(a_j, a_k) = i \ \wedge \ \forall k \geqslant \ell, f(a_j, a_k) = 1 - i\big)$

Indeed, for any $\ell > j$ such that $f(a_j, a_\ell) = i$, by semi-heredity, $f(a_j, a_{\ell-1}) = i$. So with a finite induction we get $\forall k \in \rrbracket j, \ell \rrbracket, f(a_j, a_k) = i$.

There are now two possibilities. Either there are finitely many a_j of type 2, and so by removing these elements, the resulting set is f-homogeneous for the color i. Otherwise there are infinitely many a_j of type 2, in which case one can define an infinite f-homogeneous subset for color $1 - i$ using $\mathsf{B}\Sigma^0_1(A)$ as follows: due to the observation above, "a_j is of type 2" is equivalent to the $\Sigma^0_1(A)$ formula $\exists \ell > j, f(a_j, a_\ell) = 1 - i$. Thus, given a finite set of type 2 elements $\{a_{j_0}, \dots, a_{j_{k-1}}\}$, by $\mathsf{B}\Sigma^0_1(A)$ there is $b > \max\{j_0, \dots, j_{k-1}\}$, and so j_k is defined as the smallest integer j_k such that $j_k \geqslant b$ and $f(a_{j_{k-1}}, a_{j_k}) = 1 - i$.

□

Proposition 2.6.6. $\mathsf{RCA}_0 \vdash$ CAC for trees $\implies$ SHER *and* SHER $\leqslant_c$ CAC for trees

Proof. Let $f : [\mathbb{N}]^2 \to 2$ be a semi-hereditary coloring for the color $i < 2$. We begin by constructing a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ defined as $T \coloneqq \{\sigma_n : n \in \mathbb{N}\}$, where σ_n is the unique string which is:

1. strictly increasing (as a function), with last element n
2. weak-homogeneous for the color i, i.e. $\forall k < |\sigma_n| - 1, f(\sigma_n(k), \sigma_n(k+1)) = i$
3. maximal as a weak-homogeneous set, i.e. $\forall y < \sigma_n(0), f(y, \sigma_n(0)) = 1 - i$ and $\forall j < |\sigma_n| - 1, \forall y \in \rrbracket \sigma_n(j), \sigma_n(j+1) \llbracket, f(\sigma_n(j), y) = 1 - i \vee f(y, \sigma_n(j+1)) = 1 - i$

To ensure existence, uniqueness, and that T is a tree, we prove σ_n is obtained via the following effective procedure. Start with the string n. If the string $s_0 \cdot \ldots \cdot s_m$ has been constructed, then look for the biggest integer $j < s_0$ such that $f(j, s_0) = i$. If there is none, the process ends. Else, the process is repeated with the string $j \cdot s_0 \cdot \ldots \cdot s_m$.

The string obtained is maximal by construction. It is unique, because at each step, if there are two (or more) integers $j_0 < j_1$ smaller than s_0 and such that $f(j_0, s_0) = f(j_1, s_0) = i$, then by semi-heredity we have $f(j_0, j_1) = 1$. This means we will eventually add j_0 after j_1. In particular, the string contains all the $j < n$ such that $f(j, n) = i$. Moreover, this shows T is a tree, since the procedure is the same at any point during construction.

Now we can apply CAC for trees to T, leading to two possibilities. Either there is an infinite path, which is a weakly-homogeneous set for the color i thanks to condition 2. And so apply Lemma 2.6.5 to obtain a f-homogeneous set for the color i.

Or there is an infinite antichain, which is of the form $(\sigma_{n_j})_{j\in\mathbb{N}}$. Let us show the set $H := \{n_j : j \in \mathbb{N}\}$ is f-homogeneous for the color $1-i$. Indeed, if $f(n_s, n_t) = i$ for some $s < t$, then $n_s \in \sigma_{n_t}$, since σ_{n_t} contains all the elements $y < n_t$ such that $f(y, n_t) = i$. But then $\sigma_{n_s} \prec \sigma_{n_t}$, contradicting the fact that $(\sigma_{n_j})_{j\in\mathbb{N}}$ is an antichain. $\square$

We end this section by studying RT^2_2 in relation with 3-variable forbidden patterns. As explained in the introduction, there are three basic 3-variable forbidden patterns, yielding the notions of semi-heredity, semi-ancestry, and semi-transitivity, respectively. These forbidden patterns induce Ramsey-like statements of the form "for any 2-coloring of pairs, there exists an infinite set which avoids some kind of forbidden patterns". This statement applied to semi-transitivity yields a consequence of the Erdős-Moser theorem, known to be strictly weaker than Ramsey's theorem for pairs over RCA_0. We now show that the two remaining forbidden patterns yield statements equivalent to RT^2_2. This completes the picture of the reverse mathematics of Ramsey-like theorems for 3-variable forbidden patterns.

Definition 2.6.7. A coloring $f : [\mathbb{N}]^2 \to 2$ has **semi-ancestry** for the color $i < 2$ if

$$\forall x{<}y{<}z, f(x,y) = f(x,z) = i \implies f(y,z) = i$$

Before proving RT^2_2 from the Ramsey-like statement about semi-ancestry over RCA_0, we need to prove that this statement implies $\mathsf{B}\Sigma_2$. This is done by proving the following principle.

Statement 2.6.8 (D^2_2)**.** Every Δ^0_2 set admits an infinite subset in it or its complement.

Proposition 2.6.9. *The statement "for any 2-coloring of pairs, there exists an infinite set which has semi-ancestry for some color" implies* D^2_2 *over* RCA_0.

Proof. Let A be a Δ^0_2 set whose approximations are $(A_t)_{t\in\mathbb{N}}$. We define the coloring $f(x,y) := \chi_{A_y}(x)$, and use the statement of the proposition to obtain an infinite set B that has semi-ancestry for some color.

If B has semi-ancestry for the color 1, then $\forall x<y<z \in B, x \in A_y \wedge x \in A_z \implies y \in A_z$. Now either $B \subseteq \overline{A}$ and we are done. Or $\exists x \in A \cap B$, which means $\forall^\infty y \in B, x \in A_y$, implying that $\forall^\infty y > x \in B, \forall z > y \in B, y \in A_z$ by semi-ancestry, i.e. $\forall^\infty y > x \in B, y \in A$. So we can compute a subset H of B which is infinite and such that $H \subseteq A$.

This argument also works when B has semi-ancestry for the color 0, we just need to switch A and $\overline{A}$, as well as $\in$ and $\notin$, when needed. □

Corollary 2.6.10. *The statement "for any 2-coloring of pairs, there exists an infinite set which has semi-ancestry for some color" implies* $\mathsf{B}\Sigma^0_2$ *over* RCA_0.

Proof. Immediate, since $\mathsf{RCA}_0 \vdash \mathsf{D}^2_2 \implies \mathsf{B}\Sigma^0_2$, see [CLY10, Theorem 1.4]. □

Proposition 2.6.11. *The statement "for any 2-coloring of pairs, there exists an infinite set which has semi-ancestry for some color" implies* RT^2_2 *over* RCA_0 *and over the computable reduction.*

Proof. Let $f : [\mathbb{N}]^2 \to 2$ be a coloring. We can apply the statement to obtain an infinite set A which has semi-ancestry for the color i, i.e. $\forall x<y<z \in A, f(x,y) = i \wedge f(x,z) = i \implies f(y,z) = i$. There are two possibilities. Either there exists $a \in A$ such that $\exists^\infty b > a \in A, f(a,b) = i$, in which case all such elements b form an infinite f-homogeneous set due to the property of A. Otherwise any $a \in A$ verifies $\forall^\infty b > a, f(a,b) = 1-i$, i.e. all the elements of A have a limit color equal to $1-i$ for the coloring $f\restriction_{[A]^2}$. Thus we can use $\mathsf{B}\Sigma^0_2$ (Corollary 2.6.10) to compute an infinite homogeneous set (see [DHR20, Proposition 6.2]). □

The proof that the Ramsey-like statement about semi-heredity implies Ramsey's theorem for pairs is indirect, and uses ADS.

Proposition 2.6.12. *The statement "for any 2-coloring of pairs, there exists an infinite set which is semi-hereditary for some color" implies* ADS *over* RCA_0 *and over the computable reduction.*

Proof. Let $\mathscr{L} = (\mathbb{N}, <_\mathscr{L})$ be a linear order. Let $f : [\mathbb{N}]^2 \to 2$ be the coloring defined by $f(\{x,y\}) = 1$ iff $<_\mathscr{L}$ and $<_\mathbb{N}$ coincide on $\{x,y\}$. By the statement of the proposition, there is an infinite set H on which the coloring is semi-hereditary for some color i.

Before continuing, note that if there is a pair $x < y \in H$ such that $f(x,y) = 1-i$ then $\forall z > y \in H, f(x,z) = 1-i$. Indeed, either $f(y,z) = 1-i$, in which case by transivity of $<_{\mathscr{L}}$ we have $f(x,z) = 1-i$. Or $f(y,z) = i$, in which case by semi-heredity $f(x,z) = 1-i$, because otherwise it would imply that $f(x,y) = i$.

Now there are two possibilities. Either we have $\forall^\infty x \in H, \forall y > x \in H, f(x,y) = i$, and so by getting rid of finitely many elements we obtain an infinite sequence that is increasing if $i = 1$ or decreasing if $i = 0$. Or we have $\exists^\infty x \in H, \exists y > x \in H, f(x,y) = 1-i$, in which case, by induction, we construct an infinite sequence $x_0 <_{\mathbb{N}} x_1 <_{\mathbb{N}} \ldots \in H$ such that $\forall n \in \mathbb{N}, f(x_n, x_{n+1}) = 1-i$. If $i = 0$ it will be increasing for $<_{\mathscr{L}}$, otherwise it will be decreasing for $<_{\mathscr{L}}$. The construction starts by finding $x_0, y_0 \in H$ such that $x_0 < y_0$ and $f(x_0, y_0) = 1-i$. Suppose the sequence $x_0, \ldots, x_{n-1} \in H$ has already been constructed, and that we have $y_{n-1} \in H$ such that $y_{n-1} > x_{n-1}$ and $f(x_{n-1}, y_{n-1}) = 1-i$. We then look for a $x, y \in H$ such that $y > x > y_{n-1}$ and $f(x,y) = 1-i$. By the previous remark we have that $f(x_{n-1}, x) = 1-i$, so we define $x_n \coloneqq x$ and $y_n \coloneqq y$. ☐

Corollary 2.6.13. *The statement "for any 2-coloring of pairs, there exists an infinite set which is semi-hereditary for some color" implies* RT^2_2 *over* RCA_0.

Proof. This comes from the fact that $\mathsf{RT}^2_2 \iff S + \mathsf{SHER}$ by definition (with S denoting the statement in question). And we have $\mathsf{RCA}_0 \vdash S \implies \mathsf{SHER}$ by using Proposition 2.6.12, Proposition 2.4.1 and Proposition 2.6.6. ☐

Remark 2.6.14. Let S denote the statement "for any 2-coloring of pairs, there exists an infinite set which is semi-hereditary for some color". The proof that S implies RT^2_2 over RCA_0 involves two applications of S. The first one is to obtain an infinite set over which the coloring is semi-hereditary, and the second one is to solve SHER using the fact that S implies ADS, which itself implies SHER. It is unknown whether RT^2_2 is computably reducible to S.

2.7 Stable counterparts: SADS and CAC for stable c.e. trees

Cholak, Jockusch, and Slaman [CJS01] made significant progress in the understanding of Ramsey's theorem for pairs by dividing the statement into a stable and a cohesive part. We now recall the definition of a stable coloring and extend it to linear orders.

Definition 2.7.1. A linear order $\mathscr{L} = (\mathbb{N}, <_{\mathscr{L}})$ is **stable** if it is of order type (i.e. isomorphic to) $\omega + \omega^*$.

We call SRT^2_k and SADS the restriction of RT^2_k and ADS to stable colorings and stable linear orders, respectively. Given a linear order $\mathscr{L} = (\mathbb{N}, <_{\mathscr{L}})$, the coloring corresponding to the order is stable if and only if the linear order is of order type $\omega + \omega^*$, or $\omega + k$ or $k + \omega^*$. In particular, SRT^2_2 implies SADS over RCA_0.

In this section, we study the stable counterparts of CAC for trees and SHER, to prove they are equivalent over RCA_0. We show SADS implies CAC for stable c.e. trees over RCA_0. It follows in particular that every computable instance of CAC for stable c.e. trees admits a low solution.

Definition 2.7.2 (Dorais [Dor12])**.** A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is **stable** when for every $\sigma \in T$ either $\forall^\infty \tau \in T, \sigma | \tau$ or $\forall^\infty \tau \in T, \sigma \not| \tau$

Note that any stable finitely branching tree admits a unique path.

Proposition 2.7.3. $\mathsf{RCA}_0 \vdash \mathsf{SADS} \implies$ CAC for stable c.e. trees

Remark 2.7.4. In the proof below, we use the statement that $\mathsf{RCA}_0 \vdash \mathsf{SADS} \implies \mathsf{B}\Sigma^0_2$. This is because $\mathsf{RCA}_0 \vdash \mathsf{SADS} \implies \mathsf{PART}$ ([HS07, Proposition 4.6]) and $\mathsf{RCA}_0 \vdash \mathsf{PART} \iff \mathsf{B}\Sigma^0_2$ ([CLY10, Theorem 1.2]).

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite c.e. tree that is stable.

Consider the total order $<_0$, defined by $\sigma <_0 \tau \iff \sigma \prec \tau \vee (\sigma | \tau \wedge \sigma(d) < \tau(d))$ where $d \coloneqq \min\{y : \sigma(y) \neq \tau(y)\}$. We show that it is of type $\omega + \omega^*$, i.e.

$$\forall \sigma \in T, (\forall^\infty \tau \in T \sigma <_0 \tau) \ \vee \ (\forall^\infty \tau \in T, \tau <_0 \sigma)$$

So let $\sigma \in T$, there are two possibilities. Either $\forall^\infty \tau \in T, \sigma \not| \tau$, meaning we even have $\forall^\infty \tau \in T, \sigma \prec \tau$, which directly implies $\forall^\infty \tau \in T, \sigma <_0 \tau$.

Or $\forall^\infty \tau \in T, \sigma | \tau$. In this case, we consider all the nodes τ successors of a prefix of σ but not prefix of σ, there are finitely many of them, because there are finitely many prefixes of σ and no infinitely-branching node (WLOG, as otherwise there would be a computable infinite antichain). So we can apply the pigeon-hole principle, by using $\mathsf{B}\Sigma^0_2$, to deduce that there is a certain τ which has infinitely many successors.

Moreover, by stability of T, there cannot be another such node. Indeed, by contradiction, if there were two such nodes τ and τ', then we would have $\exists^\infty \eta \in T, \eta | \tau$, because the successors of τ' are incomparable to τ. And since τ already verifies $\exists^\infty \eta \in T, \eta \succ \tau$, contradicting the stability of T.

Therefore we have that $\forall^\infty \eta \in T, \eta \succ \tau$, and so depending on whether $\tau <_0 \sigma$ or $\sigma <_0 \tau$, we obtain that $\forall^\infty \eta \in T, \eta <_0 \sigma$ or $\forall \eta \in T, \sigma <_0 \eta$ respectively. From there we can apply SADS and the proof is exactly like in Proposition 2.4.1. □

Corollary 2.7.5. CAC for stable c.e. trees *admits low solutions.*

Proof. This comes from the fact that any instance of SADS has a low solution, as proven in [HS07, Theorem 2.11]. □

The proof that SHER follows from CAC for stable trees will use $\mathsf{B}\Sigma_2$. We therefore first prove that CAC for stable trees implies $\mathsf{B}\Sigma_2$ over RCA_0.

Lemma 2.7.6. $\mathsf{RCA}_0 \vdash$ CAC for stable trees $\implies \mathsf{RT}^1_{<\infty}$

Proof. Let $f : \mathbb{N} \to k$ be a coloring. There are two possibilities: Either $\exists i < k, \exists^\infty x, f(x) = i$, in which case there is an infinite computable f-homogeneous set. Otherwise $\forall i < k, \forall^\infty x, f(x) \neq i$ and we show this leads to contradictions. Consider the infinite f-computable tree

$$T := \{\varepsilon\} \cup \left\{ \sigma \in \mathsf{Inc} : \begin{array}{l} \exists i < k, \sigma \text{ is } f\text{-homogeneous for the color } i \\ \text{and } \forall x < \max \sigma, (f(x) = i \implies x \in \operatorname{ran} \sigma) \end{array} \right\}$$

where Inc is the set of strictly increasing strings of $\mathbb{N}^{<\mathbb{N}}$.

We claim that T is a stable tree, and more precisely that $\forall \sigma \in T, \forall^\infty \tau \in T, \sigma | \tau$. Otherwise there would be some $\sigma \in T$ f-homogeneous for the color i such that $\exists^\infty \tau \in T, \sigma \prec \tau$. By definition of T, these elements τ are all f-homogeneous for the color i and form an infinite path, contradicting the assumption that $\forall^\infty x, f(x) \neq i$.

Finally, every antichain in T is of size at most k, thus T is a stable tree with no infinite path and no infinite antichain, contradicting CAC for stable trees. □

Proposition 2.7.7. *Under* RCA_0 *the statement* CAC for stable trees *implies* SHER for stable colorings

Proof. Let $f : [\mathbb{N}]^2 \to 2$ be a stable coloring, semi-hereditary for the color i. We distinguish two cases. Either there are finitely many integers with limit color i, meaning we can ignore them and use $\mathsf{B}\Sigma^0_2$ (Lemma 2.7.6) to compute an infinite homogeneous set (see [DHR20, Proposition 6.2]). Otherwise, there are infinitely many integers whose limit color is i, in which case we use the same proof as in Proposition 2.6.6, but we must prove the tree T we construct is stable. So let σ_n be a node of this tree.

Suppose first n has limit color i. Let p be sufficiently large so that $f(n,p) = i$. As explained in Proposition 2.6.6, σ_p contains all the integers $m < p$ such that $f(m,p) = i$. It follows that $n \in \sigma_p$. Moreover, if $n \in \sigma_p$, then $\sigma_n \preceq \sigma_p$. Thus, $\forall^\infty p, \sigma_n \prec \sigma_p$.

Suppose now n has limit color $1-i$, then since there are infinitely many integers with limit color i, there is one such that $p > n$. In particular, σ_p verifies $\forall^\infty \tau \in T, \sigma_p \prec \tau$. Thus, if $\sigma_n \prec \sigma_p$ then $\forall^\infty \tau \in T, \sigma_n \prec \tau$, and if $\sigma_n | \sigma_p$ then $\forall^\infty \tau \in T, \sigma_n | \tau$. □

Proposition 2.7.8. *Under* RCA$_0$ *the statement* SHER for stable colorings *implies* CAC for stable c.e. trees

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite stable c.e. tree. The proof is the same as in Proposition 2.6.3, but we must verify that the coloring $f : [\mathbb{N}]^2 \to 2$ defined is stable. Given $x \in \mathbb{N}$, we claim that $\exists i < 2, \forall^\infty y f(x,y) = i$. Since T is stable, either $\forall^\infty y, \psi(x) \not| \psi(y)$ or $\forall^\infty y, \psi(x) | \psi(y)$ holds. In the first case $\forall^\infty y, f(x,y) = 1$, and in the second one $\forall^\infty y, f(x,y) = 0$. Thus the coloring is stable, and the proof can be carried on. □

Corollary 2.7.9. *The following are equivalent over* RCA$_0$*:*

(1) CAC for stable trees
(2) CAC for stable c.e. trees
(3) SHER for stable colorings

2.8 Conclusion

The following figures sum up the relationships between the various statement explored in this paper. All implications hold both in RCA$_0$ and over the computable reduction, a red arrow means that the implication does not hold, and a double arrow means the implication is strict.

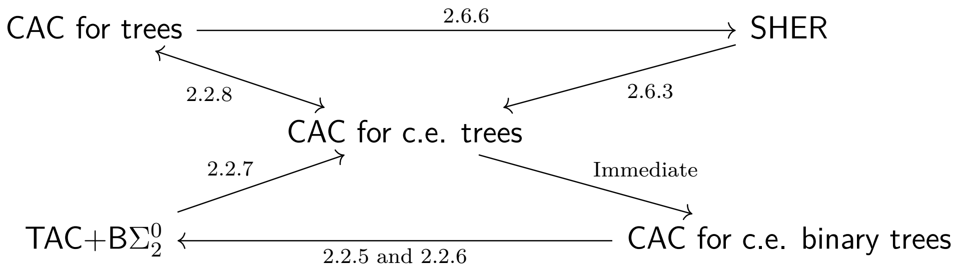

Figure 2.1: Implications proven in this chapter between CAC for trees and equivalent statements

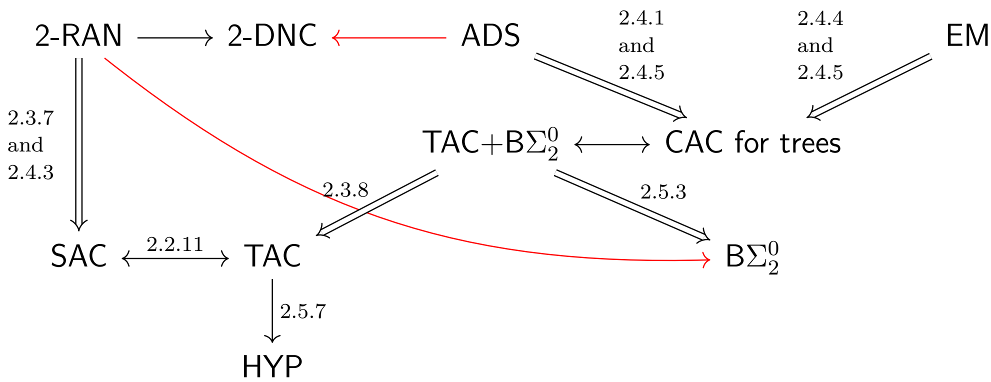

Figure 2.2: Implications between CAC for trees and the statements encoutered in this chapter

2.8.1 Open questions

We have established in Theorem 2.2.4 the equivalence between $\mathsf{TAC} + \mathsf{B}\Sigma^0_2$ and other statements.

Open question 2. What is the first-order part of TAC?

Recall that, by Corollary 2.5.3, for every fixed low set X, there is a computable instance of TAC with no X-computable solution. By computable equivalence, this property also holds for CAC for trees. It is however unknown whether Corollary 2.5.3 can be improved to defeat all low sets simultaneously.

Open question 3. Does every computable instance of CAC for trees admit a low solution?

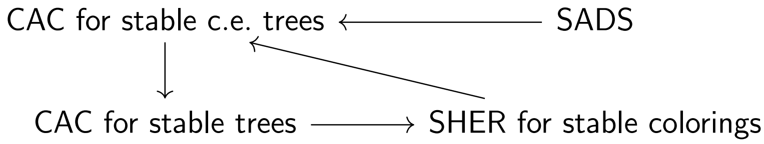

Figure 2.3: Implications between the stable version of CAC for trees and the statements encoutered in this chapter

Note that by Corollary 2.7.5, any negative witness to the previous question would yield a non-stable tree.

We have also seen by Proposition 2.3.12 that, for every computable instance T of CAC for trees, every computably bounded DNC function relative to $\varnothing'$ computes a solution to T. The natural question would be whether we can improve this result to any DNC function relative to $\varnothing'$.

Open question 4. Is there some X such that, for every computable instance T of CAC for trees, every DNC function relative to X computes a solution to T?

Note that in the case of a low set X, the answer is negative, as there exist $\mathsf{DNC}_2(X)$ functions of low degree.

Finally, recall from Remark 2.6.14 the following question:

Open question 5. Is RT^2_2 computably reducible to the statement "for any 2-coloring of pairs, there exists an infinite set which is semi-hereditary for some color"?

CHAPTER 3

CROSS-CONSTRAINT BASIS THEOREMS AND PRODUCTS OF PARTITIONS

3.1 Introduction

In this chapter, we continue the study of reductions between Ramsey statements, more precisely we focus on the case of computable reduction. We have established with Theorem 1.3.44 that RT^n_{k+1} is not computably reducible to RT^n_k. We would now like to assess whether or not RT^n_{k+1} is reducible to finitely many instances of RT^n_k. To answer this question we use a technique first developed by Liu in [Liu23] and build new basis theorems for spaces that are adapted to the study of products of problems.

3.1.1 Products of problems

There exist various operators on mathematical problems, coming essentially from the study of Weihrauch degrees. Among these, we shall consider two operators:

- The **star** of a problem P is the problem P^* whose instances are finite tuples $(X_0, \dots, X_{n-1})$ of instances of P for some $n \in \mathbb{N}$ and whose solutions are finite tuples $(Y_0, \dots, Y_{n-1})$ such that, for every $i < n$, Y_i is a P-solution to X_i.
- The **parallelization** of a problem P is the problem $\widehat{\mathsf{P}}$ whose instances are infinite sequences of P-instances $X_0, X_1, \dots$ and whose solutions are infinite

sequences $Y_0, Y_1, \dots$ such that, for every $i \in \mathbb{N}$, Y_i is a P-solution to X_i.
When considering a reduction $\mathsf{P} \leqslant_c \mathsf{Q}^*$, one is allowed to use an arbitrarily large, but finite number n of instances of Q to solve an instance X of P, where n depends on X. However, the instances of Q must be simultaneously chosen, in that they are not allowed to depend on each others' solutions. This simultaneity prevents from using the standard color blindness argument, and motivates the following question:

Question 12. Given $n, k \geqslant 1$, does $\mathsf{RT}^n_{k+1} \leqslant_c (\mathsf{RT}^n_k)^*$ hold?

The case $n = 1$ is not interesting since RT^n_k is computably true, but was studied in the context of Weihrauch degrees by Dorais et al [DDH+16], Hirschfeldt and Jockusch [HJ16], and Dzhafarov and al [DGH+20]. The first interesting case for computable reduction is then $n = 2$. Let us first illustrate why the technique used to prove that $\mathsf{RT}^2_{k+1} \not\leqslant_c \mathsf{RT}^2_k$ fails when considering products. Patey [Pat16c, Theorem 3.11] used an analysis based on preservation of hyperimmunities to prove that for every $k \geqslant 1$, every $(k+1)$-tuple of hyperimmune functions $g_0, \dots, g_k$, and every computable instance X of RT^n_k, there exists a solution Y such that at least two among the hyperimmune functions remain Y-hyperimmune, and that it is not the case for computable instances of RT^n_{k+1}. This property fails when considering the star operator, as the following lemma shows:

Lemma 3.1.1 (Cholak et al [CDHP20]). *There exist 4 hyperimmune functions $g_0, \dots, g_3$ and two computable colorings $f_0, f_1 : [\mathbb{N}]^2 \to 2$ such that for every infinite f_0-homogeneous set H_0 and every infinite f_1-homogeneous set H_1, at most one g_i is $H_0 \oplus H_1$-hyperimmune.*

Proof. Indeed, consider a Δ^0_2 4-partition $A_0 \sqcup \dots \sqcup A_3 = \mathbb{N}$ such that for every $i < 4$, $\overline{A}_i$ is hyperimmune. For every $i < 4$, g_i is the principal function of $\overline{A}_i$. Let f_0, f_1 be computable instances of RT^2_2 such that for every x, $\lim_y f_0(x,y) = 1$ iff $x \in A_0 \cup A_1$ and $\lim_y f_1(x,y) = 1$ iff $x \in A_0 \cup A_2$. Every infinite f_0-homogeneous set H_0 is either included in $A_0 \cup A_1$ or in $A_2 \cup A_3$, and every infinite f_1-homogeneous set H_1 is either included in $A_0 \cup A_2$ or in $A_1 \cup A_3$.

Note that if a set $H \subseteq A_i \cup A_j$ for some $i < j < 4$, then p_H dominates each g_k for $k \in \{0,1,2,3\} - \{i,j\}$, in which case none of those g_k are H-hyperimmune. Thus, either g_0 and g_1 or g_2 and g_3 are not H_0-hyperimmune, and either g_0 and g_2, or g_1 and g_3 are not H_1-hyperimmune. It follows that at most one of the g_i is $H_0 \oplus H_1$-hyperimmune. $\square$

For $n, k \geqslant 2$ we know that $(\mathsf{SRT}^n_k)^*$ is strictly reducible to $(\mathsf{RT}^n_k)^*$, more precisely we know that $\mathsf{RT}^n_k \not\leqslant_c (\mathsf{SRT}^n_k)^*$. Indeed, on one hand, Jockusch [Joc72b, Theorem 5.1] showed there is an instance of RT^n_k with no Σ^0_n solution, on the other hand, any instance of SRT^n_k has a Δ^0_n solution.

Liu [Liu23] proved that $\mathsf{SRT}^2_3 \not\leqslant_c (\mathsf{SRT}^2_2)^*$, answering a question by Cholak et al [CDHP20]. His proof involved completely new combinatorics, which will be presented in this chapter. We will also extend his result, give a complete answer to Question 12, and prove that $\mathsf{RT}^n_3 \not\leqslant_c (\mathsf{RT}^n_2)^*$ for every $n \geqslant 2$. Before presenting Liu's approach, note that the reduction holds when considering parallelization, but for a completely different reason.

Lemma 3.1.2. *For every $n, k \geqslant 1$, $\mathsf{RT}^{n+1}_k \leqslant_c \widehat{\mathsf{RT}^n_2}$.*

Proof. By Brattka and Rakotoniaina [BR17, Corollary 3.30], $\mathsf{WKL}^{(n)} \leqslant_c \widehat{\mathsf{RT}^n_2}$, where $\mathsf{WKL}^{(n)}$ is the problem whose instances are Δ^0_{n+1} approximations of infinite binary trees, and whose solutions are infinite paths through the trees. It follows that for every set X, there is an X-computable instance of $\widehat{\mathsf{RT}^n_2}$ such that every solution is of PA degree over $X^{(n)}$. By Cholak, Jockusch and Slaman [CJS01], for every instance X of RT^{n+1}_k, every PA degree over $X^{(n)}$ computes the jump of a solution to X. $\square$

3.1.2 Notation

In the rest of the chapter, we fix $r \in \mathbb{N}$, and use the following notations. For $N \in \mathbb{N}$ and $u \in \{\mathbb{N}, {<}\mathbb{N}, {\leqslant}N, {=}N\}$, we let $\mathcal{X}_u(0) := 3^u$, $\mathcal{X}_u(1) := (2^u)^r$ and $\mathcal{X}_u := \mathcal{X}_u(0) \times \mathcal{X}_u(1)$. For simplicity, when u is omitted it means $u := \mathbb{N}$, i.e. $\mathcal{X} := 3^{\mathbb{N}} \times (2^{\mathbb{N}})^r$.

3.2 Core ideas

In this section, we give the general picture of a proof for separating a theorem from another over computable reducibility. Then, we specialize the idea to Question 12 and explain the core ideas of Liu's technique. Note that the terminology has been freely altered from Liu's original article.

3.2.1 Separating theorems

Given two problems P and Q, to prove that $\mathsf{P} \not\leqslant_c \mathsf{Q}$, one needs to construct an instance X_P of P whose solutions are difficult to compute, and, for every X_P-computable instance X_Q of Q, a solution Y_Q to X_Q which does not X_P-compute any solution to X_P. The framework was used in its whole generality by Lerman, Solomon, and Towsner [LST13], but all the currently known separations over RCA_0 or computable reduction can be done by constructing a computable instance of P.

The construction of the instance of P can be done either by a priority construction or by an effectivization of a forcing construction. In many cases, it is constructed using the finite extension method, that is, an effectivization of Cohen forcing. This will be the case in this document as well (see Proposition 3.5.11).

Given an X_p-computable instance X_Q of Q, the solution Y_Q is usually built by forcing, in a forcing notion $(\mathbb{P}_{X_\mathsf{Q}}, \leqslant)$. The solution has to satisfy two types of properties:

- **Structural properties**: being a solution to X_Q. These properties are generally ensured by the very definition of the notion of forcing.
- **Computational properties**: not computing a solution to X_P. These properties are divided into countably many requirements, by considering each Turing functional individually. Given a requirement $\mathcal{R}_e$, one must prove that the set of conditions forcing $\mathcal{R}_e$ is dense.

There is often a tension between the structural properties which provide some computational strength, and the computational properties which require some weakness. There is however some degree of freedom in the computational properties, as they are parameterized by the instance X_P of P on which we have the hand.

The idea, coming from Lerman, Solomon and Towsner [LST13], consists in building the instance X_P considering each tuple $(X_\mathsf{Q}, c, \mathcal{R}_e)$ at a time, where X_Q is an X_p-computable instance of Q, c is a forcing condition in $(\mathbb{P}_{X_\mathsf{Q}}, \leqslant)$, and $\mathcal{R}_e$ is a requirement. Given a partial approximation of X_P and a tuple $(X_\mathsf{Q}, c, \mathcal{R}_e)$, ask whether there is an extension $d \leqslant c$ forcing $\Phi_e^{Y_\mathsf{Q}}$ to output enough bits of information. If so, complete X_P so that it diagonalizes against the functional. Otherwise, there is an extension $d \leqslant c$ forcing $\Phi_e^{Y_\mathsf{Q}}$ to be partial. The counter-intuitive part of this approach is that the satisfaction of the requirements is ensured by the construction of X_P instead of the construction of the solutions Y_Q to X_P-computable instances of Q.

As explained by Patey [Pat17], one can polish the previous construction, by abstracting the construction steps of X_Q to consider every operation with the same definitional properties, yielding some kind of genericity property. For example, the separation of the Erdős-Moser theorem and the Ascending Descending Sequence from Ramsey's theorem for pairs [LST13] were polished in [Pat17] and [Pat18] to obtain hyperimmunity and dependent hyperimmunity, respectively. In this document, the polishing step yields Γ-hyperimmunity (see Definition 3.5.9).

Separating Ramsey-type theorems

In the particular case of Ramsey-type theorems, there exists a well-established sub-scheme of construction. Many Ramsey-type theorems are of the form "For every k-coloring of the n-tuples of an infinite structure, there exists an infinite isomorphic substructure over which all the n-tuples satisfy some properties". In the case of Ramsey's theorem, the infinite structure is $(\mathbb{N}, <)$, and the property is homogeneity, but one can consider weaker properties, such as transitivity, in which case one obtains the Erdős-Moser theorem. One can also consider tree structures, yielding the tree theorem [CHM09] or Milliken's tree theorem [AdCD⁺24]. These theorems are usually proven by induction, by constructing so-called pre-homogeneous substructures. In the case of Ramsey's theorem:

Definition 3.2.1. A set $H \subseteq \mathbb{N}$ is **pre-homogeneous** for a coloring $f : [\mathbb{N}]^{n+1} \to k$ if $\forall \vec{x} \in [H]^n, \forall z > y > \max \vec{x}, f(\vec{x}, y) = f(\vec{x}, z)$

Although pre-homogeneity is the natural notion to consider from a combinatorial viewpoint, the computability-theoretic practice has shown the utility of the weaker notion of cohesiveness. It establishes a bridge between computable instances of RT^{n+1}_k and Δ^0_2 instances of RT^n_k, and can be seen as some sort of "delayed pre-homogeneity". Indeed, given a coloring $f : [\mathbb{N}]^{n+1} \to k$, one can consider the sequence of sets $\vec{R} := \langle R_{\vec{x},z} : \vec{x} \in [\mathbb{N}]^n, z < k \rangle$ defined by $R_{\vec{x},z} := \{z \in \mathbb{N} : f(\vec{x}, z) = y\}$. Thus, for an infinite $\vec{R}$-cohesive set C, the coloring f restricted to $[C]^{n+1}$ is stable. This induces a $\Delta^0_2(C)$ coloring $\widehat{f} : [C]^n \to k$.

Cohesiveness has almost no computational power. Indeed, by delaying pre-homogeneity, the statement becomes about the jump of sets. More precisely, COH is computably equivalent to the statement "For every Δ^0_2 infinite binary tree, there exists a Δ^0_2 path" (see Belanger [Bél22]). Most of the properties used to separate Ramsey-type statements are preserved by COH. This phenomenon can be explained by the fact that every set can be made Δ^0_2 without affecting too much the ground model (see Towsner [Tow15]). This will also be the case in this chapter with Theorem 3.5.16. Because of this, Question 12 (for $n = 2$) becomes a question about separating Δ^0_2 instances of RT^1_{k+1} from finite products of Δ^0_2 instances of RT^1_k.

More generally, given two Ramsey-type statements P^n, Q^n parameterized by the dimension of the n-tuples, the question of $\mathsf{P}^{n+1} \leqslant_c \mathsf{Q}^{n+1}$ is often reduced to the corresponding question with Δ^0_2 instances of P^n and Q^n. The experience shows that almost all the known separations consist in actually constructing a Δ^0_2 instance of P^n which defeats not only all the Δ^0_2 instances of Q^n, but *all* the instances of Q^n, with no effectiveness restriction (see [Pat16c, Pat16b] for examples). The previous remark about Towsner's work shows that this apparently stronger diagonalization is often without loss of generality. This will also be the case in this chapter, and the Δ^0_2 instance of RT^1_{k+1} will defeat all the finite products of instances of RT^1_k (Theorem 3.3.11).

Building a single instance of RT^1_{k+1} which defeats simultaneously uncountably many instances of $(\mathsf{RT}^1_k)^*$ raises new difficulties, as the sequence of all tuples $(X_{\mathsf{Q}}, c, \mathcal{R}_e)$ is not countable anymore. Thankfully, we shall see that there exists a single countable notion of forcing $(\mathbb{P}, \leqslant)$ such that $\mathbb{P}_X \subseteq \mathbb{P}$ for every Q-instance X. Moreover, given a condition $c \in \mathbb{P}$, the class $I(c)$ of all Q-instances X such that $c \in \mathbb{P}_X$ is a compact class. One will exploit this compactness to defeat all Q-instances $X \in I(c)$ simultaneously.

3.2.2 Cross-constraint techniques

The setting is therefore the following: in order to prove that $\mathsf{P}^{n+1} \not\leqslant_c \mathsf{Q}^{n+1}$, one builds a Δ^0_2 instance X of P^n such that, for every instance $\widetilde{X}$ of Q^n, there is a Q^n-solution $\widetilde{Y}$ to $\widetilde{X}$ which does not compute any P^n-solution to X.

The instance X of P^n is built by an effectivization of Cohen forcing. For example to prove that $\mathsf{RT}^2_{k+1} \not\leqslant_c (\mathsf{RT}^2_k)^*$, we will build a Δ^0_2 instance f of RT^1_{k+1} using an increasing sequence of $(k+1)$-valued strings $\sigma_0 \prec \sigma_1 \prec \dots$ and let f be the limit of this sequence.

Let $(\mathbb{P}, \leqslant)$ be a countable notion of forcing used to build solutions to every instance of Q^n. At stage s, assuming the Cohen condition σ_s has been defined, consider the next pair $(c, \mathcal{R}_e)$ where $c \in \mathbb{P}$ and $\mathcal{R}_e$ is a requirement saying that Φ^G_e is not a solution to the P^n-instance. Recall that $I(c)$ is the class of all Q-instances X such that $c \in \mathbb{P}_X$, and consider the class $\mathcal{C} \subseteq \operatorname{dom} \mathsf{P}^n \times I(c)$ of all pairs $(X, \widetilde{X})$ such that there is an extension $d \leqslant c$ with $\widetilde{X} \in I(c)$ forcing Φ^G_e to be partial or a P^n-solution to X. There are two cases:

- Case 1: the class $\mathcal{C}$ is **left-full** below σ_s, that is, for every instance X of P^n extending σ_s, there exists a Q^n-instance $\widetilde{X}$ such that $(X, \widetilde{X}) \in \mathcal{C}$. Then, by some appropriate basis theorem which depends on the combinatorics of P^n and Q^n, there exist multiple pairs $(X_0, \widetilde{X}_0), \dots, (X_{k-1}, \widetilde{X}_{k-1})$ in $\mathcal{C}$ such that $X_0, \dots, X_{k-1}$ are **incompatible**, in the sense that there is no set which is a solution to all these P^n-instances simultaneously, while $\widetilde{X}_0, \dots, \widetilde{X}_{k-1}$ are compatible as Q^n-instances. Then, by building a solution to the Q^n-instance which will be simultaneously a solution to $\widetilde{X}_0, \dots, \widetilde{X}_{k-1}$, this forces Φ_e^G to be partial, hence to satisfy $\mathcal{R}_e$.
- Case 2: the class $\mathcal{C}$ is not left-full below σ_s. Then, there exists a P^n-instance X extending σ_s such that, for every Q^n-instance $\widetilde{X} \in I(c)$, c forces Φ_e^G not to be a P^n-solution to X. By compactness of $I(c)$, an initial segment $\sigma_{s+1} \prec X$ is sufficient to witness this diagonalization, hence to satisfy $\mathcal{R}_e$.

The general idea of cross-constraint techniques takes its roots in Liu's proof of separation of Ramsey's theorem for pairs from weak König's lemma [Liu12], in a slightly different setting. Indeed, P^{n+1} was WKL, which is known to admit a maximally difficult instance, so only Q^n was built. In that article, he considered the class $\mathcal{C}$ of all pairs $(f, \widetilde{X})$ such that f is a partial function with finite support, and $\widetilde{X} \in I(c)$ is an instance of RT^1_2.

3.3 General framework

In this section, we define the fundamental notions of left-full cross-tree and prove the main theorems parameterized by the cross-constraint ideals. The basis theorems proven in Section 3.4 will show the existence of cross-constraint ideals with various computability-theoretic properties, and will be used to answer the main question in Section 3.5.3.

3.3.1 Cross-trees

When considering Π^0_1-classes for the space $\mathcal{X}$, it is natural to consider cross-trees which play a role analogous to binary trees in the case of the Cantor space.

We extend the **prefix relation** on strings $\preccurlyeq$ to tuples of strings, in the natural way. More precisely, for any $n \in \mathbb{N}$, given integers $k_0, \dots, k_{n-1}$, and two tuples $\sigma := (\sigma_0, \dots, \sigma_{n-1}), \tau := (\tau_0, \dots, \tau_{n-1}) \in \prod_{i<n} k_i^{<\mathbb{N}}$, we have $\sigma \preccurlyeq \tau$ if and only if

$\forall i < n, \sigma_i \preccurlyeq \tau_i$. For any $k \in \mathbb{N}$, the **empty string** of $k^{<\mathbb{N}}$ is denoted by ε, and we abuse the notation to also denote any tuple $(\varepsilon, \dots, \varepsilon)$.

Moreover we define its **cylinder** by $[\sigma] \coloneqq [\sigma_0] \times \dots \times [\sigma_{n-1}]$, and its **length** by $|\sigma| \coloneqq \max\{|\sigma_i| : i < n\}$.

Definition 3.3.1. A class $\mathcal{P} \subseteq \mathcal{X}$ is Π^0_1 if there is a c.e. set $W \subseteq \mathcal{X}_{<\mathbb{N}}$ such that

$$\overline{\mathcal{P}} = \bigcup_{\chi \in W} [\chi]$$

A **cross-tree** is a set $T \subseteq \mathcal{X}_{<\mathbb{N}}$ which is downward-closed for the prefix relation $\preccurlyeq$, and such that $\forall (\rho, \sigma) \in T, \forall i{<}j{<}r, |\sigma_i| = |\sigma_j|$ and $|\sigma| \leqslant |\rho|$. The **height** of T is $h(T) \coloneqq \max\{|\chi| : \chi \in T\}$

The class of its (infinite) **paths** is defined as

$$[T] \coloneqq \left\{(X, Y) \in \mathcal{X} : \forall n, (X{\upharpoonright}_n, Y{\upharpoonright}_n) \in T\right\}$$

Moreover, given a string $\rho \in \mathcal{X}_{<\mathbb{N}}(0)$, we define the tree $T[\rho] \coloneqq \{\sigma \in \mathcal{X}_{<\mathbb{N}}(1) : (\rho, \sigma) \in T\}$, which is finite since $|\sigma| \leqslant |\rho|$. A cross-tree T is said to be **right-pruned** if $\forall \rho \in \mathcal{X}_{<\mathbb{N}}(0), T[\rho]$ is pruned, i.e. all the leaves of $T[\rho]$ have length $|\rho|$. Finally, for any $N \in \mathbb{N}$ we define $T{\upharpoonright}_N \coloneqq \{\chi \in T : |\chi| \leqslant N\}$.

Lemma 3.3.2. *A class $\mathcal{P} \subseteq \mathcal{X}$ is Π^0_1 if and only if there is a computable cross-tree $T \subseteq \mathcal{X}_{<\mathbb{N}}$ such that $[T] = \mathcal{P}$.*

Proof. Let $T \subseteq \mathcal{X}_{<\mathbb{N}}$ be a computable cross-tree. The set $[T]$ is Π^0_1 as its complement is the set $\bigcup_{\chi \in W} [\chi]$ where $W \coloneqq \mathcal{X}_{<\mathbb{N}} - T$ is computable.

Now let $\mathcal{P}$ be a Π^0_1 class whose complement is $\bigcup_{(\rho,\sigma) \in W} [\rho] \times [\sigma]$. Consider the cross-tree T such that $(\rho, \sigma) \in T \iff \forall \mu \preccurlyeq \rho, \forall \tau \preccurlyeq \sigma, (\mu, \tau) \notin W[|\rho|]$ and $|\rho| \geqslant |\sigma|$. It is computable and its paths are exactly the elements of $\mathcal{P}$. □

Remark 3.3.3. In the rest of the chapter, every notion or proposition related to a class $\mathcal{P} \subseteq \mathcal{X}$, also holds for a computable cross-tree $T \subseteq \mathcal{X}_{<\mathbb{N}}$ instead, by considering its associated class $[T]$.

3.3.2 Left-fullness

In this section, we define a notion of largeness such that, any Π^0_1-class that satisfies it contains multiple members satisfying some constraints. A key factor also lies in the fact that the complexity of this notion is only Π^0_1.

Definition 3.3.4. A class $\mathcal{P} \subseteq \mathcal{X}$ is **left-full** below $(\rho, \sigma) \in \mathcal{X}_{<\mathbb{N}}$ if

$$\forall X \in [\rho], \exists Y \in [\sigma], (X, Y) \in \mathcal{P}$$

Moreover, for any integer $N \in \mathbb{N}$, we say that a finite tree $T \subseteq \mathcal{X}_{\leqslant N}$ is **left-full** below $(\rho, \sigma) \in T$ if $\forall(\mu, \tau) \in \ell(T), |\mu| = N$, and for every $\mu \in \mathcal{X}_N(0)$ extending ρ, there is some $\tau \in \mathcal{X}_N(1)$ extending σ, such that $(\mu, \tau) \in T$.
We simply say "left-full" to signify "left-full below $(\varepsilon, \varepsilon)$".

The above definition for finite trees is motivated by the following lemma. In particular it shows that T is left-full below (ρ, σ) if and only if for any $N \in \mathbb{N}$, $T{\upharpoonright}_N$ is left-full below (ρ, σ).

Lemma 3.3.5. *Let $(\rho, \sigma) \in \mathcal{X}_{<\mathbb{N}}$, and $\mathcal{P} \subseteq \mathcal{X}$ be a Π^0_1 class, whose associated computable cross-tree is T. The statement*

$$\mathcal{P} \text{ is left-full below } (\rho, \sigma) \tag{a}$$

is equivalent to

$$\forall \mu \succcurlyeq \rho, \exists \tau \succcurlyeq \sigma, |\tau| = |\mu| \text{ and } (\mu, \tau) \in T \tag{b}$$

Moreover, if T is right-pruned, then the statement is equivalent to

$$\forall \mu \succcurlyeq \rho, (\mu, \sigma) \in T \tag{c}$$

Proof. We first show (a) $\implies$ (b). Let $\mu \succcurlyeq \rho$. Consider $X \in [\mu] \subseteq [\rho]$. By (a), there is $Y \in [\sigma]$ such that $(X, Y) \in \mathcal{P}$. Thus, in particular, we have that $(\mu, Y{\upharpoonright}_{|\mu|}) \in T$. Hence $\tau \coloneqq Y{\upharpoonright}_{|\mu|}$ is the desired witness.

For the converse, let $X \in [\rho]$. We want to find $Y \in [\sigma]$ such that $(X, Y) \in \mathcal{P}$. Consider the set $S \coloneqq \{\tau \succcurlyeq \sigma : (X{\upharpoonright}_{|\tau|}, \tau) \in T\}$. Since T is a cross-tree, S is a finitely-branching cross-tree, with root σ. Moreover, it is infinite because of (b). Thus, by König's lemma, there is $Y \in (2^{\mathbb{N}})^r$ such that $\forall \ell, (X{\upharpoonright}_\ell, Y{\upharpoonright}_\ell) \in T$. Hence $(X, Y) \in \mathcal{P}$.

Last, we show (c) $\iff$ (b). For all $\mu \succcurlyeq \rho$, if there is $\tau \succcurlyeq \sigma$ such that $|\tau| = |\mu|$ and $(\mu, \tau) \in T$, then in particular $(\mu, \sigma) \in T$ since T is a cross-tree. As for the converse, consider $\mu \succcurlyeq \rho$. We have $(\mu, \sigma) \in T$, and since the cross-tree

$\{\tau : (\mu, \tau) \in T\}$ is pruned, it means there is $\tau \succcurlyeq \sigma$ in it of size $|\mu|$ and such that $(\mu, \tau) \in T$. □

The following lemma shows how left-fullness is preserved when extending or shortening the stems. Note that the first and second components do not play a symmetric role.

Lemma 3.3.6. *Let T be a computable cross-tree left-full below $(\rho, \sigma) \in \mathcal{X}_{<\mathbb{N}}$. Then*

(a) $\forall \widehat{\rho} \succcurlyeq \rho, \forall \widehat{\sigma} \preccurlyeq \sigma, T$ is left-full below $(\widehat{\rho}, \widehat{\sigma})$

(b) For every $n \geqslant |\sigma|$, there are some $\widehat{\rho} \succcurlyeq \rho$ and $\widehat{\sigma} \succcurlyeq \sigma$ such that $|\widehat{\sigma}| = n$ and T is left-full below $(\widehat{\rho}, \widehat{\sigma})$

Proof. Item (a) can be proven directly from the definition of left-full. For Item (b), consider all the extensions of σ of length n, denoted $\sigma_0, \dots, \sigma_{k-1}$ for some $k \in \mathbb{N}$, and define $\rho_0 \coloneqq \rho$. If T is left-full below (ρ_0, σ_0) then the assertion is proven. Otherwise, by Lemma 3.3.5, it means $\exists \rho_1 \succcurlyeq \rho_0, \forall \tau \succcurlyeq \sigma_0, |\tau| = |\rho_1| \implies (\rho_1, \tau) \notin T$. Now we consider the pair (ρ_1, σ_1) to see if T is left-full below it. In case it is not, use Lemma 3.3.5 as in the previous case. Proceed inductively like this for every σ_j, where $j < k$. If at some point T is left-full below (ρ_j, σ_j) then the assertion is proven. Otherwise, it means we have built a sequence of string $\rho_0 \preccurlyeq \dots \preccurlyeq \rho_k$ such that $\forall j < k, \forall \tau \succcurlyeq \sigma_j, |\tau| = |\rho_{j+1}| \implies (\rho_{j+1}, \tau) \notin T$, and since T is a tree we even have $\forall j < k, \forall \tau \succcurlyeq \sigma_j, |\tau| = |\rho_k| \implies (\rho_k, \tau) \notin T$. But the latter statement contradicts the fact that T is left-full below (ρ, σ), by considering the string ρ_k in Lemma 3.3.5. □

3.3.3 Parameterized theorems

Most of the computability-theoretic constructions of solutions to Ramsey-type theorems are done by variants of Mathias forcing, with reservoirs belonging to some Turing ideal containing only weak sets. The combinatorics of the statement usually require some closure properties on this ideal. For example, to construct solutions to computable instances of Ramsey's theorem for pairs, or to arbitrary instances of the pigeonhole principle, one requires the ideal to be a Scott ideal, that is, a model of WKL (see [DJ09, SS95]). One must then prove some basis theorem for WKL to construct Scott ideals with only weak sets.

In our case, we shall need another closure property, yielding the notion of cross-constraint ideal. The main constructions of this section will be parameterized by cross-constraint ideals, whose existence will be proven in Section 3.4.

Definition 3.3.7. Let X be an infinite set. A pair of instances (f, g) of RT^1_k is **finitely compatible on** X if for all color $i < k$ the set $X \cap f^{-1}(i) \cap g^{-1}(i)$ is finite. Whenever $X = \mathbb{N}$, we simply say that (f, g) is finitely compatible. Also, note that the negation of "finitely compatible" is "infinitely compatible"

Statement 3.3.8 (Cross-constraint principle (CC))**.** For any left-full cross-tree $T \subseteq \mathcal{X}_{<\mathbb{N}}$, there is a pair of paths $(X^i, Y^i)_{i<2}$ such that (X^0, X^1) is finitely compatible, and for all $s < r$, (Y^0_s, Y^1_s) is infinitely compatible.

The following notion of cross-constraint ideal is the equivalent of Turing ideal for CC, so $\mathcal{I}$ is an ω-model of CC.

Definition 3.3.9. A **cross-constraint ideal** is a Turing ideal $\mathcal{I} \subseteq \mathfrak{P}(\mathbb{N})$ such that, any instance $T \in \mathcal{I}$ of CC has a solution $(X^i, Y^i)_{i<2}$ such that $(X^0, Y^0) \oplus (X^1, Y^1) \in \mathcal{I}$.

Lastly, we define a notion of hyperimmunity for k-colorings f, which is an intermediate notion between Cohen genericity and hyperimmunity. It implies in particular that for every $j < k$, $\{x : f(x) \neq j\}$ is hyperimmune, but in a dependent way.

Definition 3.3.10. An instance f of RT^1_k is **hyperimmune** relative to $D \subseteq \mathbb{N}$ if for every D-computable sequence of k-tuples $((F_{n,0}, \dots, F_{n,k-1}))_{n\in\mathbb{N}}$ of mutually disjoint finite sets such that $\min \bigcup_{j<k} F_{n,j} > n$, there is $m \in \mathbb{N}$ such that $\forall j < k, F_{m,j} \subseteq f^{-1}(j)$

In other words, an instance f is hyperimmune relative to D if, for every D-computable sequence $g_0, g_1, \dots$ of partial k-valued functions with finite support, such that $\forall n \in \mathbb{N}, \min \operatorname{dom} g_n > n$, then f is a completion of some g_n. We are now ready to prove our first parameterized theorem.

Theorem 3.3.11 (Liu [Liu23, Theorem 2.1])**.** *Let $\mathcal{M}$ be a countable cross-constraint ideal and let $f \in \mathcal{X}(0)$ be hyperimmune relative to every element of $\mathcal{M}$, then for any $g \in \mathcal{X}(1)$ there is a solution $\vec{G}$ of g which, for any $Z \in \mathcal{M}$, does not Z-compute any solution of f.*

Proof. The set $\vec{G}$ is constructed by a variant of Mathias forcing, using conditions of the form $\big((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{A}\big)$ where

- $\vec{F_\alpha}$ is an r-tuple of finite sets g-homogeneous for the colors α, i.e. $\forall s < r, F_{\alpha,s} \subseteq g_s^{-1}(\alpha(s))$
- $\vec{A}$ is an r-tuple of infinite sets in $\mathcal{M}$ such that $\forall \alpha \in 2^r, \forall s < r, \min A_s > \max F_{\alpha,s}$

The idea is that we do not know in advance what the colors of homogeneity will be for the solution being constructed, so we build all the possibilities in parallel, with α indicating the colors, i.e. for any $i < r$, the set $F_{\alpha,i}$ is g_i-homogeneous for the color $\alpha(i)$.

A condition $\big((\vec{E_\alpha})_{\alpha \in 2^r}, \vec{B}\big)$ **extends** another $\big((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{A}\big)$ if, for every $\alpha \in 2^r$ and every $s < r$, we have $E_{\alpha,s} \supseteq F_{\alpha,s}$, $B_s \subseteq A_s$, and $E_{\alpha,s} - F_{\alpha,s} \subseteq A_s$.

Every sufficiently generic filter $\mathcal{F}$ for this notion of forcing induces a family of sets $(G^{\mathcal{F}}_{\alpha,s})_{\alpha \in 2^r, s<r}$ defined by

$$G^{\mathcal{F}}_{\alpha,s} = \bigcup \Big\{ F_{\alpha,s} : \big((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{A}\big) \in \mathcal{F} \Big\}$$

Given an initial segment $F_{\alpha,s}$ of a condition $\big((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{A}\big)$, it is not necessarily possible to find an extension $\big((\vec{E_\alpha})_{\alpha \in 2^r}, \vec{B}\big)$ with $|E_{\alpha,s}| > |F_{\alpha,s}|$, since it might be the case that $A_s \cap g_s^{-1}(\alpha(s))$ is empty. Thus, for any sufficiently generic filter $\mathcal{F}$, the set $G^{\mathcal{F}}_{\alpha,s}$ might not be infinite. However, there must necessarily exist some $\alpha \in 2^r$ such that $G^{\mathcal{F}}_{\alpha,s}$ is infinite for every $s < r$. Given a condition $c = \big((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{A}\big)$ and a coloring $h \in \mathcal{X}(1)$, define the set

$$V_h(c) := \{\alpha \in 2^r : \forall s < r, A_s \cap h_s^{-1}(\alpha(s)) \neq \varnothing\}$$

of "valid" combinations. Note that if $d \leqslant c$, then $V_h(d) \subseteq V_h(c)$. Moreover, $V_g(c) \neq \varnothing$ for every condition c. A "valid" combination of a condition c allows us to find an extension, as the following lemma shows.

Lemma 3.3.12. *For all conditions $c := \big((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{A}\big)$, all $\beta \in V_g(c)$ and all $s < r$, there is an extension $\big((\vec{E_\alpha})_{\alpha \in 2^r}, \vec{B}\big)$ such that $|E_{\beta,s}| > |F_{\beta,s}|$.*

Proof. Let $\beta \in V_g(c)$ and $s < r$. By definition of β consider $n \in A_s \cap g_s^{-1}(\alpha(s))$. Define $(\vec{E_\alpha})_{\alpha \in 2^r}$ to be equal to $(\vec{F_\alpha})_{\alpha \in 2^r}$, except for $E_{\beta,s} := F_{\beta,s} \cup \{n\}$. Accordingly, define $\vec{B}$ to be equal to $\vec{A}$, except for $B_s := A_s - \{0, 1, \dots, n\}$. Then $\big((\vec{E_\alpha})_{\alpha \in 2^r}, \vec{B}\big)$ is a condition which satisfies the lemma. □

Fix an enumeration of Turing functionals $(\Psi_e)_{e \in \mathbb{N}}$. For any $e \in \mathbb{N}$, $j < 3$ and $Z \in \mathcal{M}$ let

$$\mathcal{R}^Z_{e,j} := \Psi_e^{Z \oplus \vec{G}} \text{ is not an infinite subset of } f^{-1}(j)$$

Lemma 3.3.13. *For any 2^r-tuple of integers $(e_\alpha)_{\alpha \in 2^r}$, any 2^r-tuple of colors $(u_\alpha)_{\alpha \in 2^r}$ (where $u_\alpha \in 3$), any $Z \in \mathcal{M}$, and any condition c, there is an extension forcing $\bigvee_{\alpha \in V_g(c)} \mathcal{R}^Z_{e_\alpha, u_\alpha}$.*

Proof. Let $c := \big((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{A}\big)$ be a condition, and define $U(c)$ to be the $\Pi^0_1(\vec{A})$-class whose elements are colorings $\widetilde{g} \in \mathcal{X}(1)$ such that

$$\forall s < r, \forall j < 2, (A_s \cap g_s^{-1}(j) = \varnothing \implies A_s \cap \widetilde{g}_s^{\,-1}(j) = \varnothing)$$

in other words $U(c) := \{\widetilde{g} : V_{\widetilde{g}}(c) \subseteq V_g(c)\}$. Note that it is non-empty since $g \in U(c)$. This ensure that the $\widetilde{g}$ we consider later will have the same behavior as g regarding $\vec{A}$. Moreover, given $\alpha \in 2^r$, an r-tuple of finite sets $\vec{G}$ **satisfies** $(\vec{F_\alpha}, \vec{A})$ if $\forall s < r, G_s \supseteq F_{\alpha,s}$ and $G_s - F_{\alpha,s} \subseteq A_s$

For every $n \in \mathbb{N}$ consider the class $\mathcal{Q}_n$ whose elements are colorings $\widetilde{f} \in \mathcal{X}(0)$ such that

$$\begin{aligned}
&\exists \widetilde{g} \in U(c),\\
&\forall \alpha \in V_g(c), \forall \vec{G} \text{ satisfying } (\vec{F_\alpha}, \vec{A})\\
&\text{if } \vec{G} \text{ is } \widetilde{g}\text{-homogeneous for the colors } \alpha,\\
&\text{then } \Psi_{e_\alpha}^{Z \oplus \vec{G}} \cap \rrbracket n, +\infty \llbracket \subseteq \widetilde{f}^{-1}(u_\alpha)
\end{aligned}$$

Note that $\mathcal{Q}_n$ is a $\Pi^0_1(\vec{A})$ class uniformly in n. Indeed, the above formula is $\Pi^0_1(\vec{A})$, because by compactness, the existence of $\widetilde{g}$ is equivalent to finding an approximation in $\mathcal{X}_m(1)$ for any length $m \in \mathbb{N}$. Moreover, the set $V_g(c)$ depends on g which might be of arbitrary complexity, but since it is finite, it does not affect the complexity of the formula.

Case 1, $\exists n, \mathcal{Q}_n = 3^{\mathbb{N}}$. Thus fix such an n and consider the class $\mathcal{P} := \{(\widetilde{f}, \widetilde{g}) \in \mathcal{X} : \widetilde{g} \text{ is a witness of } \widetilde{f} \in \mathcal{Q}_n\}$. Since $\mathcal{P}$ is a left-full $\Pi^0_1(\vec{A})$-class and $\mathcal{M}$ is a cross-constraint ideal, there are paths $(X^i, Y^i)_{i<2} \in \mathcal{P}^2$ such that

- (X^0, X^1) is finitely compatible
- for any $s < r$, (Y^0_s, Y^1_s) is infinitely compatible on A_s
- $(X^0, Y^0) \oplus (X^1, Y^1) \in \mathcal{M}$

From the second item, define $\beta \in 2^r$ to be colors such that $\forall s < r, (Y^0_s)^{-1}(\beta(s)) \cap (Y^1_s)^{-1}(\beta(s)) \cap A_s$ is infinite. Note that $\beta \in V_{Y_0}(c) \cap V_{Y_1}(c)$. Since $Y^0, Y^1 \in U(c)$ (they witness $X^0, X^1 \in \mathcal{Q}_n$ respectively), then $V_{Y_0}(c) \cap V_{Y_1}(c) \subseteq V_g(c)$, so $\beta \in V_g(c)$. For each $s < r$, let $B_s = (Y^0_s)^{-1}(\beta(s)) \cap (Y^1_s)^{-1}(\beta(s)) \cap A_s$, and define $\vec{B} := (B_0, \dots, B_{r-1})$. We claim that $((\vec{F_\alpha})_{\alpha \in 2^r}, \vec{B})$ is the extension we are looking for. Indeed, since $(X^i, Y^i)_{i<2} \in \mathcal{P}^2$, then for every $\vec{G}$ compatible with $(F_\beta, \vec{B})$, we have $\Psi^{Z \oplus \vec{G}}_{e_\beta} \cap]\!]n, +\infty[\![\subseteq {X_0}^{-1}(u_\beta) \cap {X_1}^{-1}(u_\beta)$. Since ${X_0}^{-1}(u_\beta) \cap {X_1}^{-1}(u_\beta)$ is finite, the requirement $\mathcal{R}^Z_{e_\beta, u_\beta}$ is satisfied, so $\bigvee_{\alpha \in V_g(c)} \mathcal{R}^Z_{e_\alpha, u_\alpha}$ is satisfied as well.

Case 2, $\forall n, \mathcal{Q}_n \neq 3^{\mathbb{N}}$.

In which case, for each $n \in \mathbb{N}$, there is some $f_n \notin \mathcal{Q}_n$ and $\ell_n \in \mathbb{N}$ such that

$$\begin{aligned}
&\forall \widetilde{g} \in U(c),\\
&\exists \beta \in V_g(c), \exists \vec{H} \text{ satisfying } (\vec{F_\beta}, \vec{A}),\\
&\vec{H} \text{ is } \widetilde{g}\text{-homogeneous for the colors } \beta\\
&\text{ and } \Psi^{Z \oplus \vec{H}}_{e_\beta} \cap]\!]n, \ell_n[\![\not\subseteq f_n^{-1}(u_\beta)
\end{aligned}$$

This implies that there is an $\vec{A}$-computable sequence of 3-tuples $((E_{n,0}, E_{n,1}, E_{n,2}))_{n \in \mathbb{N}}$ of mutually disjoint finite sets such that

$$\begin{aligned}
&\forall \widetilde{g} \in U(c),\\
&\exists \beta \in V_g(c), \exists \vec{H} \text{ satisfying } (\vec{F_\beta}, \vec{A}),\\
&\vec{H} \text{ is } \widetilde{g}\text{-homogeneous for the colors } \beta\\
&\text{ and } \Psi^{\vec{A} \oplus \vec{H}}_{e_\beta} \cap]\!]n, \ell_n[\![\not\subseteq E_n(u_\beta)
\end{aligned}$$

Indeed, for each n, the set $]\!]n, \ell_n[\![$ can be partitioned into $(\{m : f_n(m) = j\})_{j<3}$, so it is sufficient to search for a coloring satisfying the above properties, as the search must end.

The coloring f is hyperimmune relative to $\vec{A}$, because $\vec{A} \in \mathcal{M}$. Hence there is some $n \in \mathbb{N}$ such that $\forall j < 3, E_{n,j} \subseteq f^{-1}(j)$. Moreover $E_{n,j} = f^{-1}(j) \cap]\!]n, \ell_n[\![$. So, by considering g, we have

$$\begin{aligned}
&\exists \beta \in V_g(c), \exists \vec{H} \text{ satisfying } (\vec{F_\beta}, \vec{A}),\\
&\vec{H} \text{ is } g\text{-homogeneous for the colors } \beta\\
&\text{ and } \Psi^{Z \oplus \vec{H}}_{e_\beta} \cap]\!]n, \ell_n[\![\not\subseteq f^{-1}(u_\beta)
\end{aligned}$$

Finally, the extension we are looking for is $\big(\{\vec{F}_\alpha\}_{\alpha\neq\beta\in 2^r}\cup\{\vec{H}\}, \vec{A}-\{0,\dots,\max\vec{H}\}\big)$. This completes the proof of Lemma 3.3.13. □

Let c_0 be a condition such that $V_g(c_0)$ is minimal for inclusion. Let $V \coloneqq V_g(c_0)$. Let $\mathcal{F}$ be a sufficiently generic filter containing c_0. In particular, $V = V_g(c)$ for every $c \in F$. By Lemma 3.3.12, for every $\alpha \in V$ and every $s < r$, $G^{\mathcal{F}}_{\alpha,s}$ is infinite. Moreover, by Lemma 3.3.13, for every $Z \in \mathcal{M}$, for every 2^r-tuple of integers $(e_\alpha)_{\alpha\in 2^r}$ and every 2^r-tuple of colors $(u_\alpha)_{\alpha\in 2^r}$, $\vec{G}^{\mathcal{F}}$ satisfies $\bigvee_{\alpha\in V}\mathcal{R}^Z_{e_\alpha,u_\alpha}$. By a pairing argument, for every $Z \in \mathcal{M}$, there is some $\alpha \in V$ such that $Z\oplus\vec{G}^{\mathcal{F}}_\alpha$ does not compute any infinite f-homogeneous set. Since $\mathcal{M} = \{Z_0, Z_1, \dots\}$ is countable, by the infinite pigeonhole principle, there exists some $\alpha \in V$ such that for infinitely many $n \in \mathbb{N}$, $Z_0 \oplus \dots \oplus Z_n \oplus \vec{G}^{\mathcal{F}}_\alpha$ does not compute any infinite f-homogeneous set. By downward-closure of this property under the Turing reduction, it holds for every n. This completes the proof of Theorem 3.3.11. □

Our first parameterized theorem has applications in terms of strong non-reducibility between non-computable instances of RT^1_{k+1} and $(\mathsf{RT}^1_k)^*$ (Theorem 3.5.19) and non-reducibility between computable instances of SRT^2_{k+1} and $(\mathsf{SRT}^2_k)^*$ (Theorem 3.5.20). We now prove a second parameterized theorem which enables us to prove separations between computable instances of Ramsey's theorem for pairs. Note that in the following theorem, the colorings $g_0, \dots, g_{r-1}$ are required to belong to $\mathcal{M}$, contrary to Theorem 3.3.11.

Theorem 3.3.14. *Let $\mathcal{M}$ be a countable cross-constraint ideal such that $\mathcal{M} \models$ COH and let $f : \mathbb{N} \to 3$ be hyperimmune relative to any element of $\mathcal{M}$, then for any $r \in \mathbb{N}$ and any $g_0, \dots, g_{r-1} : [\mathbb{N}]^2 \to 2$ in $\mathcal{M}$, there is an infinite g_i-homogeneous sets G_i for every $i < r$, such that $\bigoplus_{i<r} G_i$ does not compute any infinite f-homogeneous set.*

Proof. First, consider the sequence of sets $\vec{R} \coloneqq (R_{x,j,i})_{x\in\mathbb{N},j<2,i<r}$ defined by $R_{x,j,i} \coloneqq \{y : g_i(x,y) = j\}$. This sequence is in $\mathcal{M}$, because $\forall i < r, g_i \in \mathcal{M}$. And since $\mathcal{M} \models \mathsf{COH}$, there is an infinite $\vec{R}$-cohesive set $C \coloneqq \{c_0 < c_1 < \dots\} \in \mathcal{M}$. By choice of $\vec{R}$, for each $i < r$, the coloring $g_i{\restriction}_{[C]^2}$ is stable. Indeed, for $x \in \mathbb{N}$ and $i < r$, there is exactly one $j < 2$ for which $C \subseteq^* R_{x,j,i}$, and so this implies that $\lim_{y\in C} g_i(x,y) = j$.

Now, for $i < r$, let $h_i : \mathbb{N} \to 2, n \mapsto \lim_m g_i(c_n, c_m)$, and $\vec{h} \coloneqq (h_0, \dots, h_{r-1})$. By applying Theorem 3.3.11 to the hyperimmune function f and the colorings $\vec{h} \in \mathcal{X}(1)$, there are infinite $\vec{h}$-homogeneous sets $\vec{H}$ such that, for any $Z \in \mathcal{M}$,

$\vec{H} \oplus Z$ does not compute an infinite f-homogeneous set. In particular this is true for $Z := C \oplus \bigoplus_{i<r} g_i$, and since for any $i < r$, the set $H_i \oplus C \oplus g_i$ computes an infinite g_i-homogeneous set G_i, we deduce that $\bigoplus_{i<r} G_i$ does not compute any infinite f-homogeneous set. □

3.4 Cross-constraint basis theorems

As mentioned before, the two main theorems of Section 3.3 are parameterized by cross-constraint ideals, which are themselves built using iterated applications of the cross-constraint principle (CC). In this section, we prove various basis theorems for CC, namely, the Δ^0_2, low, cone avoidance, and non-Σ^0_1 preservation basis theorems. The Δ^0_2 and cone avoidance basis theorems for CC were previously proven by Liu [Liu23], but we give a new proof of the cone avoidance basis theorem which more similar to its classical counterpart for Π^0_1 classes.

3.4.1 Conditions

The most famous basis theorems for Π^0_1 classes are all proven using effective versions of forcing with binary trees. Similarly, all the basis theorems for CC in this section will be proven with effective variants of the same notion of forcing that we now describe.

Definition 3.4.1. For $k \in \mathbb{N}$, $\rho^0, \rho^1 \in k^{<\mathbb{N}}$, and $\mu^0, \mu^1 \in k^{\leqslant\mathbb{N}}$ such that $\forall i < 2, \rho^i \preccurlyeq \mu^i$. We say that (μ^0, μ^1) is **completely compatible** (respectively **completely incompatible**) over (ρ^0, ρ^1), if $\forall i \in [\![n, m]\!], \mu^0(i) = \mu^1(i)$ (respectively $\forall i \in [\![n, m]\!], \mu^0(i) \neq \mu^1(i)$), where $n := \min\{|\rho^0|, |\rho^1|\}$ and $m := \min\{|\mu^0|, |\mu^1|\}$. In both cases, if $n = 0$ we simply say **completely (in)compatible**.

Definition 3.4.2. A **condition-tuple** for a class $\mathcal{P} \subseteq \mathcal{X}$ is a tuple $(\rho^i, \sigma^i)_{i<2} \in \mathcal{X}^2_{<\mathbb{N}}$ such that $\mathcal{P}$ is left-full below (ρ^i, σ^i) for both $i < 2$, and $|\rho^0| = |\rho^1|$. A condition-tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2}$ for $\mathcal{Q} \subseteq \mathcal{X}$ **extends** another $(\rho^i, \sigma^i)_{i<2}$ for $\mathcal{P} \subseteq \mathcal{X}$, written $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \leqslant (\rho^i, \sigma^i)_{i<2}$, if

1. $\mathcal{Q} \subseteq \mathcal{P}$
2. for both $i < 2$, $(\widehat{\rho}^i, \widehat{\sigma}^i) \succcurlyeq (\rho^i, \sigma^i)$
3. $(\widehat{\rho}^0, \widehat{\rho}^1)$ is completely incompatible over (ρ^0, ρ^1)

Remark 3.4.3. A pair (ρ^i, σ^i) of a condition-tuple is seen as a finite approximation of some $(X^i, Y^i) \in \mathcal{P}$ we wish to build, i.e. of an element in $\mathcal{P} \cap [(\rho^i, \sigma^i)]$, such that (X^0, X^1) is completely incompatible over (ρ^0, ρ^1).

3.4.2 Cross-constraint Δ^0_2 basis theorem

The following Δ^0_2 basis theorem is an effective analysis of the simplest known combinatorial proof of the cross-constraint problem. It was proven by Liu [Liu23, Lemma 2.2]. We provide the original proof as a warmup for the next basis theorems and for the sake of completeness.

Theorem 3.4.4 (Liu [Liu23, Lemma 2.2])**.** *Any left-full computable instance T of* CC *has a solution $(X^i, Y^i)_{i<2}$ such that $(X^0, Y^0) \oplus (X^1, Y^1) \leqslant_T \varnothing'$.*

Proof. Let $\mathcal{P} := [T]$. We build a $\varnothing'$-computable sequence of condition-tuples $(\rho^i_0, \sigma^i_0)_{i<2} \geqslant (\rho^i_1, \sigma^i_1)_{i<2} \geqslant \dots$, such that, for $i < 2$ the functions $X^i := \bigcup_t \rho^i_t$ and $Y^i := \bigcup_t \sigma^i_t$ are the desired witnesses. To simplify notation, we simply say condition for condition-tuple.

Given $s < r$ and a condition $(\rho^i_t, \sigma^i_t)_{i<2}$, we want to extend the latter to $(\rho^i_{t+1}, \sigma^i_{t+1})_{i<2}$ such that $(\sigma^0_{t+1,s}, \sigma^1_{t+1,s})$ is not completely incompatible over $(\sigma^0_{t,s}, \sigma^1_{t,s})$.

If there is always such an extension, then the construction can be completed. By Lemma 3.3.5, this search requires the use of $\varnothing'$ to check whether $\mathcal{P}$ is left-full below $(\rho^i_{t+1}, \sigma^i_{t+1})$ or not (for each i).

Otherwise, there is some $s < r$ and some condition $(\rho^i_t, \sigma^i_t)_{i<2}$, such that, for any extension $(\rho^i_{t+1}, \sigma^i_{t+1})_{i<2}$, we have that $(\sigma^0_{t+1,s}, \sigma^1_{t+1,s})$ is completely incompatible over $(\sigma^0_{t,s}, \sigma^1_{t,s})$. In which case we say that $(\rho^i_t, \sigma^i_t)_{i<2}$ **excludes component** $s < r$. Nevertheless, by considering the following lemma, the construction can be completed.

Lemma 3.4.5 (Liu [Liu23, Claim 2.3])**.** *If a condition $(\rho^i, \sigma^i)_{i<2}$ excludes component $s < r$, then, for all $i < 2$, there is a coloring $Y^i_s \in 2^{\mathbb{N}}$ such that, for every $(\widehat{\rho}^i, \widehat{\sigma}^i) \succcurlyeq (\rho^i, \sigma^i)$, if $\mathcal{P}$ is left-full below $(\widehat{\rho}^i, \widehat{\sigma}^i)$ then $\forall j \in [\![n, m[\![, \widehat{\sigma}^i_s(j) = Y^i_s(j)$, where $n := |\sigma^i_s|$ and $m := |\widehat{\sigma}^i_s|$*

Proof. Fix $(\rho^i, \sigma^i)_{i<2}$ excluding component $s < r$ and fix $i < 2$.

Suppose first that for cofinitely many $j \in \mathbb{N}$, there exists a $k_j < 2$ such that for every $(\widehat{\rho}^i, \widehat{\sigma}^i) \succcurlyeq (\rho^i, \sigma^i)$ for which $\mathcal{P}$ is left-full, if $|\widehat{\sigma}^i_s| > j$ then $\widehat{\sigma}^i_s(j) = k_j$. Then let $Y^i_s(j) := k_j$ for j big enough, and let $Y^i_s(j)$ be of arbitrary value otherwise. This is well-defined since by Lemma 3.3.6, there exist suitable extensions of every length.

Suppose now that for infinitely many $j \in \mathbb{N}$, for every $k < 2$, there is a $(\widehat{\rho}, \widehat{\sigma}) \succcurlyeq (\rho^i, \sigma^i)$ for which $\mathcal{P}$ is left-full and such that $\widehat{\sigma}_s(j) \neq k$. Fix $j > |\sigma^i_s|$, and for each $k < 2$, let $(\rho_k, \sigma_k) \succcurlyeq (\rho^i, \sigma^i)$ be an extension such that $\sigma_{k,s}(j) \neq k$.

Since we are working in a 3-valued realm, there exists a $\widetilde{\rho} \succcurlyeq \rho^{1-i}$ of length greater than $\max(|\rho_0|, |\rho_1|)$ such that both $(\widetilde{\rho}, \rho_0)$ and $(\widetilde{\rho}, \rho_1)$ are completely incompatible over (ρ^0, ρ^1). Pick any pair $(\widehat{\rho}^{1-i}, \widehat{\sigma}^{1-i}) \succcurlyeq (\widetilde{\rho}, \sigma^{1-i})$ for which $\mathcal{P}$ is left-full and such that $|\widehat{\sigma}^{1-i}_s| > n$, this is possible by Lemma 3.3.6. Let $k := \widehat{\sigma}^{1-i}_s(n)$. Then $(\widehat{\rho}^{1-i}, \widehat{\sigma}^{1-i}, \rho_k, \sigma_k)$ is an extension contradicting the fact that $(\rho^i, \sigma^i)_{i<2}$ excludes component $s < r$ □

If at some point in the construction the condition $(\rho^i_t, \sigma^i_t)_{i<2}$ excludes the components of $I \subseteq r$, then we restart the construction from the begining with $(\rho^i_0, \sigma^i_0)_{i<2} := (\rho^0_t, \sigma^0_t, \rho^0_t, \sigma^0_t)$. The lemma ensures that (Y^0_s, Y^1_s) will not be finitely compatible for all $s \in I$, no matter the conditions selected for the sequence. And since r is a standard integer, the construction can only be restarted a finite number of times. □

Remark 3.4.6. The fact that X^i is an instance of RT^1_3 whereas each Y^i_s is an instance of RT^1_2 is exploited in the proof of Lemma 3.4.5, to ensure the existence of a 3-valued string $\widetilde{\rho}$ completely incompatible with two other strings simultaneously.

Corollary 3.4.7. *The class of all arithmetic sets is a cross-constraint ideal.*

3.4.3 Combinatorial lemmas

All the remaining basis theorems will involve some kind of first-jump control. They will require a much more involved combinatorial machinery that we now develop. These combinatorics are all due to Liu [Liu23], with a slightly different organization and terminology.

Definition 3.4.8. For $k \in \mathbb{N}$ and $m \leqslant n \in \mathbb{N}$, given chains $\rho^0, \rho^1 \in k^{\leqslant m}$, a total function $\varphi : k^m \to k^n$ **preserves incompatibility over** (ρ^0, ρ^1) if

1. $\forall \mu \in k^m, \varphi(\mu) \succcurlyeq \mu$
2. For all $\mu^0, \mu^1 \in k^m$ such that $(\mu^0, \mu^1) \succcurlyeq (\rho^0, \rho^1)$, if (μ^0, μ^1) is completely incompatible over (ρ^0, ρ^1) then $(\varphi(\mu^0), \varphi(\mu^1))$ is completely incompatible over (μ^0, μ^1)

Remark 3.4.9. Note that for $\widehat{\rho}^0, \widehat{\rho}^1 \in k^{\leqslant n}$ extending ρ^0, ρ^1 respectively, if φ preserves incompatibility over (ρ^0, ρ^1) then it also preserves incompatibility over $(\widehat{\rho}^0, \widehat{\rho}^1)$.

The following two lemmas are purely technical ones, used only locally to obtain the main combinatorial lemmas of this section.

Lemma 3.4.10 (Liu [Liu23, Lemma 4.3])**.** *For any $n \leqslant n' \leqslant m \in \mathbb{N}$, $\widehat{\rho} \in 3^m$ extending $\rho \in 3^n$, and any map $\psi : 3^n \to 3^{n'}$ preserving incompatibility over $(\varepsilon, \varepsilon)$ such that $\psi(\rho) \preccurlyeq \widehat{\rho}$, there is a map $\varphi : 3^n \to 3^m$ extending ψ and preserving incompatibility over $(\varepsilon, \varepsilon)$, such that $\varphi(\rho) = \widehat{\rho}$.*

Proof. To give an intuition, if we just wanted to show the existence of a map $\varphi : 3^n \to 3^m$ preserving incompatibility over $(\varepsilon, \varepsilon)$, we could have taken the function which maps $\eta \in 3^n$ to $\eta \cdot a^{m-n}$ where $a := \eta(0)$.

Fix ρ, $\widehat{\rho}$ and ψ as in the statement of the lemma, and let τ be such that $\psi(\rho)\cdot\tau = \widehat{\rho}$. For every $a < 3$, let $\tau_a \in 3^{m-n'}$ be defined by $\tau_a(x) = \tau(x) + a - \rho(0) \mod 3$. In particular, $\tau_{\rho(0)} = \tau$ and for every $a \neq b$, τ_a and τ_b are completely incompatible over $(\varepsilon, \varepsilon)$.

Let φ be the function $\varphi' \circ \psi$ where φ' maps $\eta \in 3^{n'}$ to $\eta \cdot \tau_{\eta(0)}$. Note that $\varphi' \circ \psi(\rho) = \psi(\rho) \cdot \tau_{\rho(0)} = \widehat{\rho}$. Moreover, if $\mu, \nu \in 3^{n'}$ are completely incompatible, then $\psi(\mu)(0) \neq \psi(\nu)(0)$, hence $\tau_{\mu(0)}$ and $\tau_{\nu(0)}$ are completely incompatible. It follows that $\varphi(\mu) = \psi(\mu)\cdot\tau_{\mu(0)}$ and $\varphi(\nu) = \psi(\nu)\cdot\tau_{\nu(0)}$ are completely incompatible themselves. $\square$

Lemma 3.4.11 (Liu [Liu23, Lemma 4.4]). *For $n, N \in \mathbb{N}$ and $f : A \to \mathfrak{P}\big(\mathcal{X}_N(1)\big)$ a total function that is order-reversing for $\subseteq$, where $A \coloneqq 3^{<\mathbb{N}} - 3^{<n}$. There is $m \in \mathbb{N}$ and a map $\varphi : 3^n \to 3^m$ preserving incompatibility over $(\varepsilon, \varepsilon)$, such that $\forall \rho \in 3^n, \forall \nu \in \mathcal{X}_N(1)$ either $\forall \widehat{\rho} \succcurlyeq \varphi(\rho), \nu \in f(\widehat{\rho})$ or $\nu \notin f(\varphi(\rho))$.*

Proof. We construct a finite sequence of integers $(n_s)_{s\leqslant p}$, for some $p \in \mathbb{N}$, along with a finite sequence of maps $(\varphi_s : 3^{n_0} \to 3^{n_s})_{s\leqslant p}$ preserving incompatibility over $(\varepsilon, \varepsilon)$.

For each $\nu \in \mathcal{X}_N(1)$ and $\rho \in 3^n$ there is a step to ensure that the map φ we construct satisfies the requirement

$$\mathcal{R}_{\nu,\rho} \coloneqq \forall \widehat{\rho} \succcurlyeq \varphi(\rho), \nu \in f(\widehat{\rho}) \text{ or } \nu \notin f(\varphi(\rho))$$

Let $n_0 \coloneqq n$ and $\varphi_0 : 3^{n_0} \to 3^{n_0}$ be the identity function. At step $s < p$, consider $\nu \in \mathcal{X}_N(1)$ and $\rho \in 3^n$. If $\forall \widehat{\rho} \succcurlyeq \varphi_s(\rho), \nu \in f(\widehat{\rho})$ we are done by defining $n_{s+1} \coloneqq n_s$ and $\varphi_{s+1} \coloneqq \varphi_s$. Otherwise there is η extending $\varphi_s(\rho)$ such that $\nu \notin f(\eta)$. Define $n_{s+1} \coloneqq |\eta|$, and use Lemma 3.4.10 to find a map $\varphi_{s+1} : 3^n \to 3^{n_{s+1}}$ extending φ_s, preserving incompatibility over $(\varepsilon, \varepsilon)$, and such that $\varphi_{s+1}(\rho) = \eta$. So, under the assumption made on f we have that $\forall \widehat{\eta} \succcurlyeq \eta, f(\widehat{\eta}) \subseteq f(\eta)$, and thus $\forall \widehat{\eta} \succcurlyeq \eta, \nu \notin f(\widehat{\eta})$. Then as φ_{s+1} extends φ_s, $\mathcal{R}_{\nu,\rho}$ is fulfilled.

Finally, define $m \coloneqq n_p$ and $\varphi \coloneqq \varphi_p$. By the hypothesis made on the function f, for every $\nu \in \mathcal{X}_N(1)$, if $\nu \in f(\varphi(\rho))$ for some $\rho \in 3^n$, then $\forall \widehat{\rho} \succcurlyeq \varphi(\rho), \nu \in f(\widehat{\rho})$ □

The following definition should be understood in the light of the first-jump control for Π^0_1 classes. When trying to construct an infinite path through an infinite binary tree $T \subseteq 2^{<\omega}$, one must ensure that at every step, the node is **extensible**, that is, the branch below the node is infinite. Being extensible is a Π^0_1 property, and therefore to obtain good first-jump control, one must resort to an overapproximation: Given a set $A \subseteq 2^{<\omega}$, instead of asking whether there is an extensible node in $T \cap A$, one will ask whether there is a level in the tree such that every node at this level belongs to A. Among the nodes at that level, at least one must be extendible. If A is Σ^0_1, then the former question is Σ^0_2, while the latter is Σ^0_1.

One can use a different technique, and ask whether the Π^0_1 class $\mathcal{P}$ of infinite subtrees of T disjoint from A is empty. In particular, by considering $S \subseteq T$, the pruned subtree of T containing only extendible nodes, since $S \notin \mathcal{P}$, there is an extendible node in $S \cap A$. Here again, this overapproximation is Σ^0_1.

In the case of cross-constraint problems, a node (ρ, σ) is extensible in a cross-tree T if T is left-full below (ρ, σ). The notion of T-sufficiency below is therefore the counterpart of the Σ^0_1 question above, mutatis mutandis.

Definition 3.4.12 (Liu [Liu23, Definition 4.22])**.** Given a tree $T \subseteq \mathcal{X}$, and a tuple $(\rho^i, \sigma^i)_{i<2} \in \mathcal{X}^2_{<\mathbb{N}}$, a set $A \subseteq \mathcal{X}^2_{<\mathbb{N}}$ is **T-sufficient over** $(\rho^i, \sigma^i)_{i<2}$ if, for every infinite subtree $S \subseteq T$ for which $(\rho^i, \sigma^i)_{i<2}$ is a condition-tuple, there is a tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in A \cap S^2$ such that $\forall i < 2, (\widehat{\rho}^i, \widehat{\sigma}^i) \succcurlyeq (\rho^i, \sigma^i)$, and $(\widehat{\rho}^0, \widehat{\rho}^1)$ is completely incompatible over (ρ^0, ρ^1).

Remark 3.4.13. Note that $(\rho^i, \sigma^i)_{i<2}$ is not a condition-tuple for T, because T is not necessarily left-full below it. However, when it is the case, we will be able to find extensions in A for which T is still left-full, see Lemma 3.4.15.

Note that if A is Σ^0_1, then the statement "A is T-sufficient over $(\rho^i, \sigma^i)_{i<2}$" is $\Sigma^0_1(T)$. The combinatorics for the cross-constraint problem are more complicated than the ones for weak König's lemma, and one cannot simply consider the pruned tree containing only extensible nodes. However, the following lemmas show that one can consider a weakly pruned tree in which every node is almost extensible, in the sense that every node can be extended into a node below which the cross-tree is left-full.

The following lemma uses compactness to give a finite cross-tree version of T-sufficiency. Recall that the notion of left-fullness was extended to finite trees, which induces a notion of condition-tuple.

Lemma 3.4.14. *Let $T \subseteq \mathcal{X}$ be a cross-tree. If $A \subseteq \mathcal{X}^2_{<\mathbb{N}}$ is T-sufficient over $(\rho^i, \sigma^i)_{i<2} \in \mathcal{X}^2_{<\mathbb{N}}$, then there is $N \in \mathbb{N}$ such that for every finite cross-subtree $S \subseteq T \cap \mathcal{X}_{\leqslant N}$ for which $(\rho^i, \sigma^i)_{i<2}$ is a condition-tuple, there is a tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in A \cap S^2$ such that $\forall i < 2, (\widehat{\rho}^i, \widehat{\sigma}^i) \succcurlyeq (\rho^i, \sigma^i)$, and $(\widehat{\rho}^0, \widehat{\rho}^1)$ is completely incompatible over (ρ^0, ρ^1).*

Proof. Consider the class $\mathcal{T}$ of finite subtrees $S \subseteq T$ whose leaves are all of the same length, such that S is left-full below (ρ^i, σ^i) for both $i < 2$, and such that for all tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in S^2$ which extends (ρ^i, σ^i) and such that $(\widehat{\rho}^0, \widehat{\rho}^1)$ is completely incompatible over (ρ^0, ρ^1), then $(\widehat{\rho}^0, \widehat{\rho}^1) \notin A$. It forms a tree for the relation where $S_0 \leqslant S_1$ if and only if $S_1{\upharpoonright}_{h(S_0)} = S_0$.

There is no infinite path in $\mathcal{T}$, otherwise it would contradict the assumption that A is T-sufficient. Hence $\mathcal{T}$ is finite thanks to König's lemma. In other words, there is $N \in \mathbb{N}$ such that, for any finite subtree $S \subseteq T \cap \mathcal{X}_{\leqslant N}$ left-full below both (ρ^i, σ^i), there is $(\widehat{\rho}^i, \widehat{\sigma}^i) \in A \cap S^2$ witnessing the T-sufficiency of A. □

The following lemma is the desired combinatorial lemma.

Lemma 3.4.15 (Liu [Liu23, Subclaim 4.7])**.** *Let $T \subseteq \mathcal{X}$ be a cross-tree that is left-full below $(\rho^i, \sigma^i)_{i<2} \in \mathcal{X}^2_{<\mathbb{N}}$. If $A \subseteq \mathcal{X}^2_{<\mathbb{N}}$ is T-sufficient over $(\rho^i, \sigma^i)_{i<2}$ and closed under extension[a] then there is a condition-tuple in A which extends $(\rho^i, \sigma^i)_{i<2}$.*

[a] That is to say, if $(\tau^i, \nu^i)_{i<2} \in A$ and $\forall i < 2, (\widehat{\tau}^i, \widehat{\nu}^i) \succcurlyeq (\tau^i, \nu^i)$, then $(\widehat{\tau}^i, \widehat{\nu}^i)_{i<2} \in A$

Proof. Let $N \in \mathbb{N}$ witness Lemma 3.4.14, and for both $i < 2$ let $A_i \coloneqq 3^{<\mathbb{N}} - 3^{N-|\rho^i|}$ For $i < 2$, define the maps

$$\begin{aligned} f_i : A_i &\to \mathfrak{P}\big(\mathcal{X}_{N-|\sigma^i|}(1)\big) \\ \tau &\mapsto \big\{\nu : (\rho^i \cdot \tau, \sigma^i \cdot \nu) \in T\big\} \end{aligned}$$

And consider the map $f : \tau \mapsto f_0(\tau) \cup f_1(\tau)$.

Since T is left-full below (ρ^i, σ^i) for both $i < 2$, we have $\forall \tau, f(\tau) \neq \varnothing$. Indeed, consider $i < 2$, by left-fullness, there is $\widetilde{\nu}$ such that $|\sigma^i \cdot \widetilde{\nu}| = |\rho^i \cdot \tau|$ and $(\rho^i \cdot \tau, \sigma^i \cdot \widetilde{\nu}) \in T$. Since $\tau \in A_i$, we deduce that $|\rho^i \cdot \tau| \geqslant N$, and since T is a tree, we have that $\nu \coloneqq \widetilde{\nu}\upharpoonright_{N-|\sigma^i|}$ verifies $(\rho^i \cdot \tau, \sigma^i \cdot \nu) \in T$.

Moreover, each f_i is order-preserving since T is a cross-tree, so f is also non-decreasing. Thus we can apply Lemma 3.4.11 on f to obtain $L \in \mathbb{N}$ and a map $\psi : 3^{N-|\rho^0|} \to 3^L$ preserving incompatibility over (ρ^0, ρ^1) such that, for any $\tau \in \mathcal{X}_{N-|\rho^0|}(0)$ and any $\nu \in \mathcal{X}_{N-|\sigma^i|}(1)$ either T is left-full below $(\rho^i \cdot \psi(\tau), \sigma^i \cdot \nu)$ or $(\rho^i \cdot \psi(\tau), \sigma^i \cdot \nu) \notin T$. In other word, ψ is such that

$$\big(\rho^i \cdot \psi(\tau), \sigma^i \cdot \nu\big) \in T \implies T \text{ is left-full below } \big(\rho^i \cdot \psi(\tau), \sigma^i \cdot \nu\big) \tag{3.4.1}$$

Now for each $i < 2$, define the set

$$B_i \coloneqq \big\{(\rho^i \cdot \tau, \sigma^i \cdot \nu) \in T : (\tau, \nu) \in \mathcal{X} \text{ and } (\rho^i \cdot \psi(\tau), \sigma^i \cdot \nu) \in T\big\}$$

And consider $S \subseteq T$, the downward-closure of $B_0 \cup B_1$. We claim that S is a cross-subtree of T for which $(\rho^i, \sigma^i)_{i<2}$ is a condition-tuple.

Indeed, fix some $i < 2$ and let $\tau \in 3^{N-|\rho^i|}$. Just as we did above to show that $\forall \tau, f(\tau) \neq \varnothing$, since T is left-full below (ρ^i, σ^i), there is some $\nu \in \mathcal{X}_{N-|\sigma^i|}(1)$ such that $(\rho^i \cdot \psi(\tau), \sigma^i \cdot \nu) \in T$. Thus, by 3.4.1, T is left-full below $(\rho^i \cdot \psi(\tau), \sigma^i \cdot \nu)$ and by definition of B_i, $(\rho^i \cdot \tau, \sigma^i \cdot \nu) \in B_i \subseteq S$. Thus S is left-full below (ρ^i, σ^i).

By Lemma 3.4.14, there is a tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in A \cap S^2$ such that $\forall i < 2, (\widehat{\rho}^i, \widehat{\sigma}^i) \succcurlyeq (\rho^i, \sigma^i)$, and $(\widehat{\rho}^0, \widehat{\rho}^1)$ is completely incompatible over (ρ^0, ρ^1). Since A is closed under extension, we can suppose without loss of generality that $|\widehat{\rho}^i| = |\widehat{\sigma}^i| = N$, hence that $(\widehat{\rho}^i, \widehat{\sigma}^i) \in B_i$. Let $\tau^i \in \mathcal{X}_{N-|\rho^i|}(0)$ and $\nu^i \in \mathcal{X}_{N-|\sigma^i|}(1)$ be such that $\widehat{\rho}^i = \rho^i \cdot \tau^i$ and $\widehat{\sigma}^i = \sigma^i \cdot \nu^i$. By definition of B_i, $(\rho^i \cdot \psi(\tau^i), \sigma^i \cdot \nu^i) \in T$, thus by Equation (3.4.1), T is left-full below $(\rho^i \cdot \psi(\tau^i), \sigma^i \cdot \nu^i)$. Thus, $(\rho^i \cdot \psi(\tau^i), \sigma^i \cdot \nu^i)_{i<2}$ is a condition-tuple for $[T]$. Moreover, since ψ preserves incompatibility over (ρ^0, ρ^1), then $(\rho^i \cdot \psi(\tau^i), \sigma^i \cdot \nu^i)_{i<2}$ is an extension of $(\rho^i, \sigma^i)_{i<2}$, and also $\forall i < 2, (\rho^i \cdot \psi(\tau^i), \sigma^i \cdot \nu^i) \succcurlyeq (\widehat{\rho}^i, \widehat{\sigma}^i)$, thus $\rho^i \cdot \psi(\tau^i), \sigma^i \cdot \nu^i)_{i<2} \in A$. □

3.4.4 Cross-constraint cone avoidance basis theorem

We now prove our first cross-constraint basis theorem which requires some sort of first-jump control, using the combinatorics developed in Section 3.4.3. This basis theorem was first proven by Liu [Liu23, Lemma 4.5] using a different argument. Our new proof follows more closely the standard proof of the cone avoidance basis theorem for Π^0_1 classes.

Theorem 3.4.16 (Cross-constraint cone avoidance, Liu [Liu23, Lemma 4.5])**.** *Let C be a non-computable set. Any left-full computable instance T of* CC *has a solution $(X^i, Y^i)_{i<2}$ such that $(X^0, Y^0) \oplus (X^1, Y^1) \not\geqslant_T C$.*

Proof. To prove the theorem, we use forcing with conditions of the form $((\rho^i, \sigma^i)_{i<2}, U, B)$, where

- U is a B-computable cross-subtree of T
- $(\rho^i, \sigma^i)_{i<2}$ is a condition-tuple for $[U]$
- $B \subseteq \mathbb{N}$ and $B \not\geqslant_T C$

A condition $((\mu^i, \tau^i)_{i<2}, S, A)$ **extends** another $((\rho^i, \sigma^i)_{i<2}, U, B)$ if $A \geqslant_T B$, $S \subseteq U$ and $(\mu^i, \tau^i)_{i<2}$ extends $(\rho^i, \sigma^i)_{i<2}$ as a condition-tuple.

We will satisfy the following requirements for each $e \in \mathbb{N}$:

$$\mathcal{R}_e : \Phi_e^{(X^0,Y^0)\oplus(X^1,Y^1)} \neq C$$

A condition $((\rho^i, \sigma^i)_{i<2}, U, B)$ **forces** $\mathcal{R}_e$, if $\mathcal{R}_e$ holds for all $(X^i, Y^i) \in [U]$ extending (ρ^i, σ^i) for each $i < 2$.

Lemma 3.4.17. *For every condition $c := ((\rho^i, \sigma^i)_{i<2}, U, B)$ and every $e \in \mathbb{N}$, there is an extension of c that forces $\mathcal{R}_e$.*

Proof. For all $x \in \mathbb{N}, v < 2$, define

$$A_{x,v} := \left\{(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in \mathcal{X}^2_{<\mathbb{N}} : \Phi_e^{(\widehat{\rho}^0,\widehat{\sigma}^0)\oplus(\widehat{\rho}^1,\widehat{\sigma}^1)}(x) \downarrow = v\right\}$$

The set $A_{x,v}$ is upward-closed, and Σ^0_1 uniformly in (x, v). Consider the following $\Sigma^0_1(B)$ set:

$$Q = \left\{(x, v) : A_{x,v} \text{ is } U\text{-sufficient over } (\rho^i, \sigma^i)_{i<2}\right\}$$

Case 1. $(x, C(x)) \notin Q$ for some $x \in \mathbb{N}$. Let $\mathscr{L}$ be the $\Pi^0_1(B)$ class of cross-trees $S \subseteq U$ witnessing that $A_{x,C(x)}$ is not U-sufficient over $(\rho^i, \sigma^i)_{i<2}$. By the cone avoidance basis theorem, there is some cross-tree $S \in \mathscr{L}$ such that $S \oplus B \not\geqslant_T C$. The condition $d := \left((\rho^i, \sigma^i)_{i<2}, S, S \oplus B\right)$ is the extension we are looking for. Indeed, it forces $\mathcal{R}_e$, because for $(X^i, Y^i)_{i<2} \in [d]$, if $\Phi_e^{(X^0,Y^0)\oplus(X^1,Y^1)}$ is total, then it is different from C on input x.

Case 2. $(x, 1 - C(x)) \in Q$ for some $x \in \mathbb{N}$. Unfolding the definition, $A_{x,1-C(x)}$ is U-sufficient over $(\rho^i, \sigma^i)_{i<2}$, so by Lemma 3.4.15, there is a condition-tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in A_{x,1-C(x)}$ extending $(\rho^i, \sigma^i)_{i<2}$. Thus the condition $\left((\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2}, U, B\right)$ is the desired extension, as it forces $\mathcal{R}_e$.

Case 3. Neither Case 1 nor Case 2 holds. Then Q is the $\Sigma^0_1(B)$ graph of the characteristic function of C, so $C \leqslant_T B$. Contradiction. □

We are now ready to prove Theorem 3.4.16. Let $c_0 := ((\rho^i, \sigma^i)_{i<2}, U, B)$ be a condition that excludes a maximal number of components and let $\mathcal{F}$ be a filter containing c_0 that is sufficiently generic for this notion of forcing. For $i < 2$, let $X^i := \bigcup\{\widehat{\rho}^i : ((\widehat{\rho}^i, \sigma^i)_{i<2}, U, B) \in \mathcal{F}\}$ and $Y^i := \bigcup\{\widehat{\sigma}^i : ((\rho^i, \widehat{\sigma}^i)_{i<2}, U, B) \in \mathcal{F}\}$. By Lemma 3.3.6, X^i and Y^i are both infinite sequences. By Lemma 3.4.5, for every $s < r$, $Y^0_s \cap Y^1_s$ is infinite. By Lemma 3.4.17, $(X^0, Y^0) \oplus (X^1, Y^1) \not\geqslant_T C$. This completes the proof of Theorem 3.4.16. □

Corollary 3.4.18. *For any non-computable set $C \subseteq \mathbb{N}$, there is a cross-constraint ideal that does not contain C.*

Proof. We construct a sequence of sets $Z_0 \leqslant_T Z_1 \leqslant_T \dots$ such that for any integer $n = \langle k, e\rangle$, $Z_n \not\geqslant_T C$, and if $\Phi_e^{Z_k}$ is an instance of CC, then Z_{n+1} computes a solution.

Define $Z_0 := \varnothing$. Suppose Z_n has been defined and let $n = \langle k, e\rangle$. If $\Phi_e^{Z_k}$ is not a left-full cross-tree, then $Z_{n+1} := Z_n$. Otherwise, by Theorem 3.4.16 relativized to Z_n, there is a pair of paths P_0 and P_1, such that $P_0 \oplus P_1 \oplus Z_n \not\geqslant_T C$. In which case $Z_{n+1} := P_0 \oplus P_1 \oplus Z_n$.

By construction, the class $\mathcal{M} := \{X \in 2^{\mathbb{N}} : \exists n, X \leqslant_T Z_n\}$ is a cross-constraint ideal containing only sets avoiding the cone above C, in particular C is not in the ideal. $\square$

3.4.5 Cross-constraint preservation of non-Σ^0_1 definitions

We now prove a second cross-constraint basis theorem, about preservation of non-Σ^0_1 definitions. This basis theorem for Π^0_1 classes was first proven by Wang [Wan16, Theorem 3.6], and implies the cone avoidance basis theorem in a straightforward way. Later, Downey et al. [DGHT$^+$22, Theorem 4.2] actually proved that the two basis theorems are equivalent, as any problem satisfying any of them, satisfies both. Thus, the following theorem is a non-trivial consequence of Theorem 3.4.16. However, we give a direct proof of it, to get familiar with the combinatorics of the cross-constraint problem.

Theorem 3.4.19 (Cross-constraint preservation of non-Σ^0_1 definitions)**.** *Let C be a non-Σ^0_1 set. Any computable instance T of* CC*, has a solution $(X^i, Y^i)_{i<2}$ such that C is not Σ^0_1 relative to $(X^0, Y^0) \oplus (X^1, Y^1)$.*

Proof. To prove the theorem, we use forcing with conditions of the form $((\rho^i, \sigma^i)_{i<2}, U, B)$ where

- U is a B-computable cross-subtree of T
- $(\rho^i, \sigma^i)_{i<2}$ is a condition-tuple for $[U]$
- $B \subseteq \mathbb{N}$ is such that C is not $\Sigma^0_1(B)$

A condition $((\mu^i, \tau^i)_{i<2}, S, A)$ **extends** another $((\rho^i, \sigma^i)_{i<2}, U, B)$ if $A \geqslant_T B$, $S \subseteq U$ and $(\mu^i, \tau^i)_{i<2}$ extends $(\rho^i, \sigma^i)_{i<2}$ as a condition-tuple.

We want to satisfy the following requirements for every Turing index e:

$$\mathcal{R}_e : W_e^{(X^0,Y^0)\oplus(X^1,Y^1)} \neq C$$

Lemma 3.4.20. *For every condition $c := ((\rho^i, \sigma^i)_{i<2}, U, B)$ and every $e \in \mathbb{N}$, there is an extension of c forcing $\mathcal{R}_e$.*

Proof. Given some $x \in \mathbb{N}$, consider the set

$$A_x := \{(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in \mathcal{X}^2_{<\mathbb{N}} : x \in W_e^{(\widehat{\rho}^0,\widehat{\sigma}^0)\oplus(\widehat{\rho}^1,\widehat{\sigma}^1)}\}$$

Here again, the set A_x is upward-closed and Σ^0_1 uniformly in x. Let

$$Q \coloneqq \{x : A_x \text{ is } U\text{-sufficient over } (\rho^i, \sigma^i)_{i<2}\}$$

The set Q is $\Sigma^0_1(B)$, thus $Q \neq C$. This leads to two cases.

Case 1. There is $x \in C - Q$. Let $\mathscr{L}$ be the class of all cross-trees $S \subseteq U$ which witness that A_x is not U-sufficient over $(\rho^i, \sigma^i)_{i<2}$. It is non-empty by hypothesis, and since A_x is Σ^0_1, then $\mathscr{L}$ is $\Pi^0_1(B)$. Now since WKL admits preservation of non-Σ^0_1 definitions (see [Wan16, Theorem 3.6]), there is a cross-tree $S \in \mathscr{L}$ such that C is not $\Sigma^0_1(S \oplus B)$. The condition $d \coloneqq \big((\rho^i, \sigma^i)_{i<2}, S, S \oplus B\big)$ is the extension we are looking for, since $x \in C$ but d forces that $x \notin W_e^{(X^0,Y^0)\oplus(X^1,Y^1)}$.

Case 2. There is $x \in Q - C$. Unfolding the definition, A_x is U-sufficient over $(\rho^i, \sigma^i)_{i<2}$, so by Lemma 3.4.15, there is a condition-tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in A_x$ extending $(\rho^i, \sigma^i)_{i<2}$. The condition $\big((\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2}, U, B\big)$ is the desired extension, as it forces $x \in W_e^{(X^0,Y^0)\oplus(X^1,Y^1)}$ for some $x \notin C$. □

We are now ready to prove Theorem 3.4.19. Let $c_0 \coloneqq ((\rho^i, \sigma^i)_{i<2}, U, B)$ be a condition that excludes a maximal number of components and let $\mathcal{F}$ be a filter containing c_0 that is sufficiently generic for this notion of forcing. For $i < 2$, let $X^i \coloneqq \bigcup\{\rho^i : ((\rho^i, \sigma^i)_{i<2}, U, B) \in \mathcal{F}\}$ and $Y^i \coloneqq \bigcup\{\sigma^i : ((\rho^i, \sigma^i)_{i<2}, U, B) \in \mathcal{F}\}$. By Lemma 3.3.6, X^i and Y^i are both infinite sequences. By Lemma 3.4.5, for every $s < r$, $Y^0_s \cap Y^1_s$ is infinite. By Lemma 3.4.20, C is not $\Sigma^0_1((X^0, Y^0) \oplus (X^1, Y^1))$. This completes the proof of Theorem 3.4.19. □

Corollary 3.4.21. *For any non-Σ^0_1 set $C \subseteq \mathbb{N}$. There is a cross-constraint ideal such that C is not Σ^0_1 relative to any element of the ideal.*

Proof. We construct a sequence of sets $Z_0 \leqslant_T Z_1 \leqslant_T \dots$ such that for any integer $n = \langle k, e\rangle$, C is not $\Sigma^0_1(Z_n)$, and if $\Phi_e^{Z_k}$ is an instance of CC, then Z_{n+1} computes a solution.

Define $Z_0 \coloneqq \varnothing$. Suppose Z_n has been defined, and let $n = \langle k, e\rangle$. If $\Phi_e^{Z_k}$ is not a left-full cross-tree, then $Z_{n+1} \coloneqq Z_n$. Otherwise, by Theorem 3.4.19 relativized to Z_n, there is a pair of paths P_0 and P_1, such that C is not Σ^0_1 relative to $P_0 \oplus P_1 \oplus Z_n$. In which case $Z_{n+1} \coloneqq P_0 \oplus P_1 \oplus Z_n$.

By construction, the class $\mathcal{M} \coloneqq \{X \in 2^{\mathbb{N}} : \exists n, X \leqslant_T Z_n\}$ is a cross-constraint ideal such that C is not Σ^0_1 relative to any element in it. □

Corollary 3.4.22 (Cross-constraint cone avoidance)**.** *Let C be a non-computable set. Any left-full computable instance T of CC, has a solution $(X^i, Y^i)_{i<2}$ such that $(X^0, Y^0) \oplus (X^1, Y^1) \not\geqslant_T C$.*

Proof. Suppose C is non-computable. Then either C or $\overline{C}$ is not Σ^0_1. By Theorem 3.4.19, there is a solution $(X^i, Y^i)_{i<2} \in [T]^2$ such that either C or $\overline{C}$ is not Σ^0_1 relative to $(X^0, Y^0) \oplus (X^1, Y^1))$. In particular, $(X^0, Y^0) \oplus (X^1, Y^1) \not\geq_T C$. □

3.4.6 Cross-constraint low basis theorem

The low basis theorem for Π^0_1 classes is one of the most famous theorems in computability theory. We prove its counterpart for the cross-constraint problem. However, contrary to the case of Π^0_1 classes, where the theorem can be strengthened to obtain superlow sets, it does not seem to be the case for cross-constraint problems.

Theorem 3.4.23 (Cross-constraint low basis)**.** *Any left-full computable instance T of* CC*, has a solution $(X^i, Y^i)_{i<2}$ such that $(X^0, Y^0) \oplus (X^1, Y^1)$ is low.*

Proof. To prove the theorem, we use forcing with conditions of the form

$$\big((\rho^i, \sigma^i)_{i<2}, U, B\big)$$

where

- U is a B-computable cross-subtree of T
- $(\rho^i, \sigma^i)_{i<2}$ is a condition-tuple for $[U]$
- B is a set of low degree

An **index** for a condition $((\rho^i, \sigma^i)_{i<2}, U, B)$ is a tuple $((\rho^i, \sigma^i)_{i<2}, a, b)$ such that $\Phi^B_a = U$ and $\Phi^{\varnothing'}_b = B'$. An index is therefore a finite representation of a condition. We say that a condition $c \coloneqq ((\rho^i, \sigma^i)_{i<2}, U, B)$ **decides the jump on** e if either $\Phi_e^{(\rho^0,\sigma^0)\oplus(\rho^1,\sigma^1)}(e)\downarrow$ holds or c forces $\Phi_e^{(X^0,Y^0)\oplus(X^1,Y^1)}(e)\uparrow$.

Lemma 3.4.24. *For every condition $c \coloneqq ((\rho^i, \sigma^i)_{i<2}, U, B)$ and every $e \in \mathbb{N}$, there is an extension d of c deciding the jump on e. Moreover, an index for d can be found $\varnothing'$-uniformly in e and an index for c.*

Proof. Consider the following Σ^0_1 set

$$A_e \coloneqq \{(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in \mathcal{X}^2_{<\mathbb{N}} : \Phi_e^{(\widehat{\rho}^0,\widehat{\sigma}^0)\oplus(\widehat{\rho}^1,\widehat{\sigma}^1)}(e)\downarrow\}$$

Case 1. A_e is not U-sufficient over $(\rho^i, \sigma^i)_{i<2}$. Let $\mathscr{L}$ be the class of all cross-trees $S \subseteq T$ which witness that A_e is not U-sufficient over $(\rho^i, \sigma^i)_{i<2}$. Since

A_e is Σ^0_1, then $\mathscr{L}$ is $\Pi^0_1(B)$. By the uniform low basis theorem relative to B (see [HJKH+08, Theorem 4.1]), there is some $S \in \mathscr{L}$ such that $(S \oplus B)' \leqslant_T \varnothing'$. Moreover, a lowness index of $S \oplus B$ (an integer a such that $\Phi_a^{\varnothing'} = (S \oplus B)'$) can be $\varnothing'$-computed from an index of $\mathscr{L}$. The condition $d := ((\rho^i, \sigma^i)_{i<2}, S, S \oplus B)$ is the extension we are looking for. Indeed, $\Phi_e^{(X^0,Y^0)\oplus(X^1,Y^1)}(e) \uparrow$ holds for any $(X^i, Y^i)_{i<2} \in [d]$.

Case 2. A_e **is** U**-sufficient over** $(\rho^i, \sigma^i)_{i<2}$**.** By Lemma 3.4.15, there is a condition-tuple $(\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2} \in A_e$ for $[U]$ which extends $(\rho^i, \sigma^i)_{i<2}$. Thus, the condition $\big((\widehat{\rho}^i, \widehat{\sigma}^i)_{i<2}, U, B\big)$ is the desired extension, since $\Phi_e^{(\widehat{\rho}^0,\widehat{\sigma}^0)\oplus(\widehat{\rho}^1,\widehat{\sigma}^1)}(e) \downarrow$.

Finally, note that $\varnothing'$ can decide whether or not it is in the first or second case, since A_e is Σ^0_1, and so "A_e is U-sufficient over $(\rho^i, \sigma^i)_{i<2}$" also is. Hence, each extension can be uniformly computed from $\varnothing'$. □

We are now ready to prove Theorem 3.4.23. We build a uniformly $\varnothing'$-computable descending sequence of conditions $c_0 \geqslant c_1 \geqslant \dots$ such that for every n, letting $c_n := ((\rho^i_n, \sigma^i_n)_{i<2}, U_n, B_n)$

- c_{n+1} decides the jump on n ;
- $|\rho^i_n| \geqslant n$; $|\sigma^i_{n,s}| \geqslant n$ for every $s < r$;
- $(\sigma^0_{n+1,s}, \sigma^1_{n+1,s})$ is not completely incompatible over $(\sigma^0_{n,s}, \sigma^1_{n,s})$.

Let $c_0 = ((\rho^i, \sigma^i)_{i<2}, U, B)$ be a condition that excludes a maximal number of components. Note that c_0 does not need to be found in $\varnothing'$, since it is a one-time guess. Assuming c_n has been defined, by Lemma 3.4.24, there is an extension $c^1_n \leqslant c_n$ deciding the jump on n. By Lemma 3.3.6, there is an extension $c^2_n \leqslant c^1_n$ satisfying the second item, and by Lemma 3.4.5, there is an extension $c_{n+1} \leqslant c^2_n$ satisfying the third item. Moreover, indices for each of these extensions can be found $\varnothing'$-computably uniformly in n. This completes the proof of Theorem 3.4.23. □

Corollary 3.4.25. *There is a cross-constraint ideal that contains only low sets.*

Proof. We construct a sequence of sets $Z_0 \leqslant_T Z_1 \leqslant_T \dots$ such that for any integer $n = \langle k, e\rangle$, Z_n is low, and if $\Phi_e^{Z_k}$ is an instance of CC, then Z_{n+1} computes a solution.

Define $Z_0 := \varnothing$. Suppose Z_n has been defined, and let $n = \langle k, e\rangle$. If $\Phi_e^{Z_k}$ is not a left-full cross-tree, then $Z_{n+1} := Z_n$. Otherwise, by Theorem 3.4.23 relativized to Z_n, there is a pair of paths P_0 and P_1, such that $(P_0 \oplus P_1 \oplus Z_n)' \leqslant_T Z_n'$. In which case $Z_{n+1} := P_0 \oplus P_1 \oplus Z_n$.

By construction, the class $\mathcal{M} := \{X \in 2^{\mathbb{N}} : \exists n, X \leqslant_T Z_n\}$ is a cross-constraint ideal containing only low sets. □

3.5 Products of instances for Ramsey's theorem

In this last section, we focus on the case of products of instances for Ramsey's theorem. We first define a notion developed by Liu in [Liu23], that is similar to hyperimmunity, and use it to establish a preservation result for COH. This result and the basis theorems established earlier are then applied to prove separation results.

3.5.1 Γ-hyperimmunity

We now define the notion of Γ-hyperimmunity, first introduced by Liu [Liu23, Section 4.3]. It is a generalization of hyperimmunity (see Lemma 3.5.10) that is tailored to be preserved by CC (see Theorem 3.5.13). Nonetheless, it is much more complex to define as it is based on an iterated process.

The idea behind Γ-hyperimmunity relies on keeping track of a finite list of "candidates" (potential approximations of the final 3-coloring) via some tree structures. The trees themselves will represent all the possible ways of selecting the list of candidates. The action of adding or removing candidates is controlled by the following definition.

Definition 3.5.1. A tree $T_1 \subseteq \mathbb{N}^{<\mathbb{N}}$ is a **one-step variation** of $T_0 \subseteq \mathbb{N}^{<\mathbb{N}}$ if there is a node $\xi \in T_0$ and a non-empty finite set $F \subseteq \mathbb{N}$ such that

- either $\xi \in \ell(T_0)$ and $T_1 = T_0 \cup \xi \cdot F$
- or $\xi \in T_0 - \ell(T_0)$, $T_1 = (T_0 - [\xi]^{\prec}) \cup \xi \cdot F$ and $F \subsetneq \{n \in \mathbb{N} : \xi \cdot n \in T_0\}$

In other words, a one-step variation of a tree consists in either extending a leaf with finitely many immediate children, or backtracking by removing the children of a node, except finitely many immediate ones. This is a non-reflexive relation.

The evolution of the list of candidates will correspond to a sequence of one-step variations of some trees called **Γ-spaces**. To be more precise, we use induction to define, for any $m \in \mathbb{N}$, a partial order $(\Gamma_m, \preccurlyeq_m)$ forming a tree whose root is denoted ζ_m.

Firstly, Γ_0 is the tree of depth 1 whose elements are the functions $f : \mathbb{N} \to 3$ with finite domain, the root ζ_0 is the empty map, and every other element is an immediate child of the root. Then, when the tree Γ_m is constructed for some m, the goal of the next tree Γ_{m+1} is to keep track of all the possible ways of selecting candidates in Γ_m. It does so through its structure. More formally:

Definition 3.5.2. Fix a partial order $(W, \preccurlyeq)$ which is a tree of root ζ. A **computation path** on $(W, \preccurlyeq)$ is a finite sequence $(T_0, \varphi_0), (T_1, \varphi_1), \dots, (T_{u-1}, \varphi_{u-1})$ where, for all $j < u$, $T_j \subseteq \mathbb{N}^{<\mathbb{N}}$ is a finite tree such that

- $T_0 = \{\varepsilon\}$
- $j \in \mathbb{N}$, T_{j+1} is a one-step variation of T_j

And, for all $j < u$, $\varphi_j : T_j \to W$ is a function such that

- $\varphi_j(\varepsilon) = \zeta$
- φ_j is order-preserving
- φ_{j+1} and φ_j are compatible, i.e. $\varphi_{j+1} = \varphi_j$ on the domain $T_{j+1} \cap T_j$

Hence, Γ_{m+1} is the set of computation paths on $(\Gamma_m, \preccurlyeq_m)$. Its root ζ_{m+1} is the computation path $(\{\varepsilon\}, \varepsilon \mapsto \zeta_m)$, and $\preccurlyeq_{m+1}$ is the prefix relation on sequences. Informally, in the case of Γ_1, the root is the nowhere-defined function, the immediate children are finite sets of functions with finite support, and the sub-branches consist of removing elements from this finite set.

A list of candidates can be associated to any computation path, by essentially “flattening” the trees.

Definition 3.5.3. The **interpretation** $[\![\gamma]\!]$ of a computation path $\gamma \in \Gamma_m$ is a finite non-empty subset of Γ_0 defined inductively as follows:

- if $m = 0$, $[\![\gamma]\!] = \{\gamma\}$
- if $m > 0$ and $\gamma := ((T_0, \varphi_0), \dots, (T_{u-1}, \varphi_{u-1}))$, then $[\![\gamma]\!] = \bigcup_{\zeta \in \ell(T_{u-1})} [\![\zeta]\!]$

Note that, because backtracking is a one-step variation, the interpretation may gain elements, even though the underlying tree structure is progressing.

Moreover, thanks to the next lemma, we know that “progressing on a tree” always terminates. It also gives us more information about the structure of the Γ spaces, which is reminiscent of the Hydra game.

Lemma 3.5.4 (Liu [Liu23, Lemma 4.12])**.** *For all $m \in \mathbb{N}$, the tree Γ_m is well-founded.*

Proof. By induction on m, we show that Γ_m is well-founded. For $m = 0$ the result comes from the definition of Γ_0. Now suppose the result holds for some m, and let H be the height of Γ_m, we prove the result for $m + 1$.

By contradiction, suppose there is an infinite path $(T_0, \varphi_0), (T_1, \varphi_1), \dots$ in Γ_{m+1}.

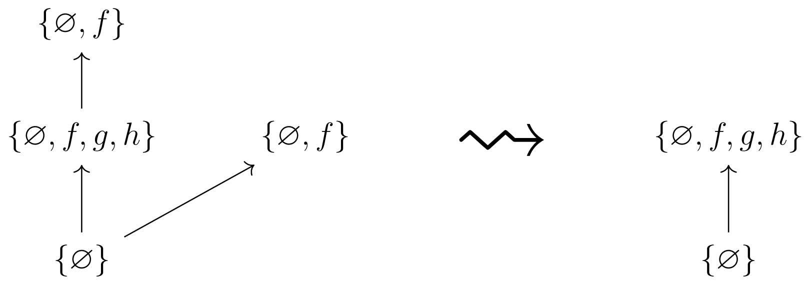

Figure 3.1: Representation of a computation path in Γ_2 gaining elements. The sets represent subtrees of Γ_0. So on the left we have a subtree of Γ_1 whose interpretation is $\{\varnothing, f\}$. The squiggly arrow represents the "or" case of a one-step variation applied to the root. And on the right we have a subtree of Γ_1 whose interpretation is $\{\varnothing, f, g, h\}$.

Claim 3.5.5. *For all $h \leqslant H$, there is $S_h \subseteq \mathbb{N}^{\leqslant h}$ and $s_h \in \mathbb{N}$ such that*

$$\forall t > s_h, \{\xi \in T_t : |\xi| \leqslant h\} = S_h$$

Proof. We prove the result by induction on h. For $h = 0$ we have $S_0 = \{\varepsilon\}$ and $s_0 = 0$, because $\varepsilon \in T_0$ and a one-step variation can only remove a node through the "or" case, which cannot apply to ε since it has no parent. Now suppose the result holds for some $h < H$. By hypothesis, the one-step variations after T_{s_h} can only apply to nodes of length greater or equal to h, otherwise the value of S_h would change, leading to a contradiction.

Consider $\xi \in T_{s_h}$ such that $|\xi| = h$, there are two possibilities. Either ξ will never have any child, i.e. $\forall t \geqslant s_h, \forall n \in \mathbb{N}, \xi \cdot n \notin T_t$, in which case we define $R_\xi := \varnothing$ and $r_\xi = s_h$. Or ξ already has or will have an immidiate successor, i.e. $\exists t \geqslant s_h, \exists n \in \mathbb{N}, \xi \cdot n \in T_t$. Since the children of ξ result from a one-step variation, there must be finitely many of them. Moreover, this number must decrease over time. Indeed, the "either" case cannot be applied to ξ anymore, as it is not a leaf anymore, and it never will be again. Because the "or" case cannot be applied to any ancestor of ξ, and if it is applied to ξ, then one of its children must remain, by definition of one-step variation. Thus, the number of children of ξ will decrease and ultimately stabilize at some point $r_\xi \in \mathbb{N}$. Let $R_\xi \subseteq \mathbb{N}^{h+1}$ be the stabilized set of children of ξ. We can now define $S_{h+1} := \bigcup_{\xi \in S_h} R_\xi$ and $s_{h+1} := \max_{\xi \in S_h} r_\xi$. They verify the desired property. □

For all $s \in \mathbb{N}$, the height of T_s is inferior or equal to H, since $\varphi_i : T_s \to \Gamma_m$ is an embedding. Thus, the previous claim implies that $\forall t \geqslant s_H, T_t = T_{s_H}$. This is a contradiction since a one-step variation of a tree is necessarily different from that tree. □

We now introduce a notion that corresponds to arrays, due to the role they play in the definition of hyperimmunity.

Definition 3.5.6. For $n \in \mathbb{N}$, a set $F \subseteq \Gamma_0$ is **over** n if for every $g \in F$, $\operatorname{dom} g \subseteq]n, \infty[$. By extension, for $m, n \in \mathbb{N}$, we say $\gamma \in \Gamma_m$ is **over** n if $[\![\gamma]\!]$ is over n.

Definition 3.5.7. For $m \in \mathbb{N}$, a Γ_m**-approximation** is a function $f : \mathbb{N}\times\mathbb{N} \to \Gamma_m$ (for some m) if, for all $n \in \mathbb{N}$, the following properties hold.

- $f(n, 0) = \zeta_m$
- $f(n, -)$ is order-preserving
- $\forall s \in \mathbb{N}, f(n, s)$ is over n

We also define its interpretation as

$$\begin{aligned} [\![f]\!] : \mathbb{N} \times \mathbb{N} &\to \mathfrak{P}_{\text{fin}}(\Gamma_0) \\ n, s &\mapsto [\![f(n, s)]\!] \end{aligned}$$

These approximations essentially list some paths from the Γ-space they are associated with. Observe that, given a Γ_m-approximation f, the limit $\lim_s f(n, s)$ exists for any integer n, because Γ_m is well-founded.

Finally, we can state the definition of Γ-hyperimmunity. Its structure is similar to that of regular hyperimmunity, "diagonal against" replaces the usual condition on the coloring, and Γ-approximations replace arrays.

Definition 3.5.8. A (potentially partial) function $f : \mathbb{N} \to 3$ is **diagonal against** a set $F \subseteq \Gamma_0$ if f extends some element in F, i.e. $\exists g \in F, \operatorname{dom} f \supseteq \operatorname{dom} g$ and $f{\restriction}_{\operatorname{dom} g} = g$. By extension, for $m \in \mathbb{N}$, a partial 3-coloring f is **diagonal against** $\gamma \in \Gamma_m$, if it is diagonal against $[\![\gamma]\!]$.

The term "diagonal against" will become clearer with the next definition, where, like in a diagonalization argument, the function f tries to defeat countably many finite sets F, whose purpose is to contain functions that are all different from f, i.e. $\forall g \in F, \exists x \in \operatorname{dom} f \cap \operatorname{dom} g, f(x) \neq g(x)$.

Definition 3.5.9. A 3-coloring f is **Γ-hyperimmune** relative to $D \subseteq \mathbb{N}$ if, for every $m \in \mathbb{N}$ and for every D-computable Γ_m-approximation g, there is an $n \in \mathbb{N}$ such that f is diagonal against $\lim_s g(n, s)$.

The next lemma shows that Γ-hyperimmunity is a stronger version of hyperimmunity.

Lemma 3.5.10. *If a 3-coloring f is Γ-hyperimmune relative to $D \subseteq \mathbb{N}$, then it is also hyperimmune relative to D.*

Proof. Let $(F_{n,0}, F_{n,1}, F_{n,2})_{n\in\mathbb{N}}$ be a D-computable sequence of mutually disjoint finite sets $(F_{n,0}, F_{n,1}, F_{n,2})_{n\in\mathbb{N}}$ such that $\min \bigcup_{j<3} F_{n,j} > n$.

For any $n \in \mathbb{N}$, consider we define a 3-coloring with finite support

$$\gamma_n : x \mapsto \begin{cases} j & \text{if } x \in F_{n,j} \\ \uparrow & \text{otherwise} \end{cases}$$

It is well defined since the finite sets $F_{n,j}$ are mutually disjoint. And consider the function

$$\begin{aligned} g : \mathbb{N}^2 &\to \Gamma_0 \\ n, s &\mapsto \begin{cases} \varnothing & \text{if } s = 0 \\ \gamma_n & \text{otherwise} \end{cases} \end{aligned}$$

It is a D-computable Γ_0-approximation, thus by Γ-hyperimmunity of f there is $N \in \mathbb{N}$ such that f is diagonal against $\lim_s g(N, s)$, i.e. f extends γ_N, i.e. $\forall j < 3, F_{N,j} \subseteq f^{-1}(j)$. □

The following proposition ensures the existence of a Γ-hyperimmune coloring.

Proposition 3.5.11 (Liu [Liu23, Lemma 4.17])**.** *There is a Δ^0_2 coloring which is Γ-hyperimmune.*

Proof. First, it is possible to computably list all computable Γ_m-approximations, where m is any integer. Indeed, given a computable partial order $(W, \preccurlyeq)$, the set of its computation paths is uniformly computable. This is because they are finite sequences composed of finite trees, so all the constraints listed in the definition can be computed. From there, the set of Γ_m-spaces is uniformly computable in m.

Hence, we fix an enumeration $(\Phi_{e,m})_{e\in\mathbb{N}}$ of all the computable Γ_m-approximations, for any m.

We wish to construct $f : \mathbb{N} \to 3$ such that, for any $e, m \in \mathbb{N}$, the following requirement is satisfied.

$$\mathcal{R}_{e,m} \coloneqq f \text{ is diagonal against } \Phi_{e,m}$$

Suppose we have so far constructed $\rho \in 3^{<\mathbb{N}}$, we now consider the Γ_m-approximation $\Phi_{e,m}$, and we are going to use $\varnothing'$ to find an integer s such that $\Phi_{e,m}(|\rho|, s)$ is the limit value for $\Phi_{e,m}(|\rho|, -)$. Then either $[\![\Phi_{e,m}(|\rho|, s)]\!] = \varnothing$, in which case $\mathcal{R}_{e,m}$ is satisfied. Or $[\![\Phi_{e,m}(|\rho|, s)]\!] \neq \varnothing$, in which case $\mathcal{R}_{e,m}$ is satisfied for $\rho \cup \tau$ where $\tau \in \Phi_{e,m}(|\rho|, s)$. Because, by definition of a Γ_m-approximation, τ is over $|\rho|$, i.e. $\min \operatorname{dom} \tau > |\rho|$

To find $s \in \mathbb{N}$, build a sequence of integers, starting with $s_0 \coloneqq 0$. If s_k is the latest integer we have defined, then use $\varnothing'$ to know whether $\exists y > s_k, \Phi_{e,m}(|\rho|, y) \succ_m \Phi_{e,m}(|\rho|, s_k)$. If the answer is yes then s_{k+1} is defined by such a witness, otherwise we stop defining the sequence and $s \coloneqq s_k$.

The above procedure must end at some point, because

$$\Phi_{e,m}(|\rho|, s_0) \prec_m \Phi_{e,m}(|\rho|, s_1) \prec_m \dots$$

is a strictly increasing sequence in the space Γ_m, which is well-founded (by Lemma 3.5.4). And so $\Phi_{e,m}(|\rho|, s)$ is the limit value we were looking for, because the case that defines s ensures that $\forall y > s, \Phi_{e,m}(|\rho|, y) = \Phi_{e,m}(|\rho|, s)$. □

In the article of Liu [Liu23], the notion of Γ-hyperimmunity yields two new basis theorems, one on regular Π^0_1 classes and the other on the cross version of Π^0_1 classes.

Theorem 3.5.12 (Liu [Liu23, Lemma 4.18]). *Let $f : \mathbb{N} \to 3$ be Γ-hyperimmune. For every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there is a member $X \in \mathcal{P}$ such that f is Γ-hyperimmune relative to X.*

Theorem 3.5.13 (Liu [Liu23, Lemma 4.2]). *Let $f : \mathbb{N} \to 3$ be Γ-hyperimmune. For every computable instance of* CC, *there is a solution X such that f is Γ-hyperimmune relative to X.*

For proofs of these theorems we refer the reader to Liu's original article, as they were already quite optimally written and our formalism would not really add any value.

3.5.2 Preservation of Γ-hyperimmunity for COH

Following the idea of decomposing RT^2_2 into $\mathsf{SRT}^2_2 + \mathsf{COH}$, we prove the following theorem of preservation for COH, as it will intervene in the proof of Theorem 3.5.20.

To later simplify diagonalization arguments, we will only consider relevant functionals, i.e. the ones that yield only Γ-approximations. The following lemma ensures that we are able to do this.

Definition 3.5.14. For any $m \in \mathbb{N}$, a Turing functional Ψ is a **Γ_m-functional** if and only if it is total and Ψ^X is a Γ_m-approximation for every oracle X.

Lemma 3.5.15. *For every $m \in \mathbb{N}$ and every Turing functional Ξ, there is a Γ_m-functional Ψ such that, for any oracle X, if Ξ^X is a Γ_m-approximation, then Ψ^X has the same limit function. Moreover, the index of Ψ is obtained uniformly from the index of Ξ.*

Proof. The functional Ψ proceeds as follows. Fix an oracle X and some $n \in \mathbb{N}$. Define $\Psi^X(n, 0) := \zeta_m$. Now to define $\Psi^X(n, t)$ for $t > 0$, consider $s < t$ the biggest integer (if it exists) such that $\Xi^X(n, s)[t]\downarrow$ and such that $\Xi^X(n, s)[t] \succ \Psi^X(n, t-1)$. If s exists then $\Psi^X(n, t) := \Xi^X(n, s)[t]$. Otherwise, $\Psi^X(n, t) := \Psi^X(n, t-1)$. $\square$

We can now proceed to the proof of the preservation result.

Theorem 3.5.16. *Let $g \in 3^{\mathbb{N}}$ be a Γ-hyperimmune function and $R_0, R_1, \dots$ be a uniformly computable sequence of sets. Then there is an $\vec{R}$-cohesive set G such that g is Γ-hyperimmune relative to G.*

Proof. We proceed by forcing, using Mathias conditions (F, X) such that X is computable. For a Γ_m-functional Ψ, define the requirement $\mathcal{R}_{\Psi,m}$:= there is $n \in \mathbb{N}$ such that g is diagonal against $\lim_s \Psi^G(n, s)$.

Lemma 3.5.17. *For each condition (F, X), $m \in \mathbb{N}$, and Γ_m-functional Ψ, there is an extension forcing $\mathcal{R}_{\Psi,m}$*

Proof. We define a computable Γ_m-approximation $f : \mathbb{N}^2 \to \Gamma_m$ as follows. First, for every n, $f(n, 0) := \zeta_m$. Suppose that at step s, we have defined $f(n, s)$ for every n. Then for each n, if there is some $F' \subseteq X$ with $\max F' < s$ and some $t \leqslant s$ such that $\Psi^{F \cup F'}(n, t)\downarrow$ and $\Psi^{F \cup F'}(n, t) \succ f(n, s)$, then let $f(n, s + 1) := \Psi^{F \cup F'}(n, t)$. Otherwise, let $f(n, s + 1) := f(n, s)$. Then go to the next stage.

By construction, f is a Γ_m-approximation. Since g is Γ-hyperimmune, there is $n \in \mathbb{N}$ such that g is diagonal against $\lim_s f(n,s)$. Now by definition of f, there is a finite (possibly empty) $F' \subseteq X$ and $t \in \mathbb{N}$ such that $\lim_s f(n,s) = \Psi^{F \cup F'}(n,t)$. We claim that $(F \cup F', X - \{0,\dots,\max F'\})$ forces $\mathcal{R}_{\Psi,m}$. Indeed, by construction of f and since we considered the limit, we have that, for every $F'' \subseteq X - \{0,\dots,\max F'\}$ and cofinitely many $t' \in \mathbb{N}$, $\Psi^{F \cup F' \cup F''}(n,t') = \Psi^{F \cup F'}(n,t)$. □

Let $\mathcal{F}$ be a sufficiently generic filter for computable Mathias forcing and let $G = \bigcup_{(F,X) \in \mathcal{F}} F$. By genericity, G is $\vec{R}$-cohesive, since given a condition (F,X) and a computable set R_x, either $(F, X \cap R_x)$ or $(F, X \cap \overline{R}_x)$ is a valid extension. By Lemma 3.5.17, for every m and Γ_m-functional Γ, g diagonalizes against Γ^G. Thus, by Lemma 3.5.15, g is Γ-hyperimmune relative to G. □

Corollary 3.5.18. *For every Γ-hyperimmune function $f : \mathbb{N} \to 3$, there exists a cross-constraint ideal $\mathcal{M} \models \mathsf{COH}$ such that f is Γ-hyperimmune relative to every element of $\mathcal{M}$.*

Proof. We construct a sequence of sets $Z_0 \leqslant_T Z_1 \leqslant_T \dots$ such that for any integer $n = \langle i,k,e \rangle$,

- f is Γ-hyperimmune relative to Z_n.
- if $n = \langle 0,k,e \rangle$, and if $\Phi_e^{Z_k}$ is an instance of CC, then Z_{n+1} computes a solution.
- if $n = \langle 1,k,e \rangle$, and if $\Phi_e^{Z_k}$ is an instance of COH, then Z_{n+1} computes a solution.

Define $Z_0 \coloneqq \varnothing$. Suppose Z_n has been defined. If $n = \langle 0,k,e \rangle$ and $\Phi_e^{Z_k}$ is not a left-full cross-tree, then $Z_{n+1} \coloneqq Z_n$. Otherwise, by Theorem 3.5.13 relativized to Z_n, there is a pair of paths P_0 and P_1, such that f is Γ-hyperimmune relative to $P_0 \oplus P_1 \oplus Z_n$. In which case $Z_{n+1} \coloneqq P_0 \oplus P_1 \oplus Z_n$. If $n = \langle 1,k,e \rangle$ and $\Phi_e^{Z_k}$ is not a countable sequence of sets, then $Z_{n+1} \coloneqq Z_n$. Otherwise, by Theorem 3.5.16 relativized to Z_n, there is a $\Phi_e^{Z_k}$-cohesive set C, such that f is Γ-hyperimmune relative to $C \oplus Z_n$. In which case $Z_{n+1} \coloneqq C \oplus Z_n$.

By construction, the class $\mathcal{M} \coloneqq \{X \in 2^{\mathbb{N}} : \exists n, X \leqslant_T Z_n\}$ is a cross-constraint ideal such that f is Γ-hyperimmune relative to every element of $\mathcal{M}$. □

3.5.3 Separation results

We now present the different reducibility results that can be obtained from the theorems established in this chapter.

Theorem 3.5.19 (Liu [Liu23, Theorem 2.1]). $\mathsf{RT}^1_3 \not\leqslant_{soc} (\mathsf{RT}^1_2)^*$

Proof. Let $\mathcal{M}$ be a countable cross-constraint ideal (such an ideal exists thanks to Corollary 3.4.7), and let $f \in \mathcal{X}(0)$ be hyperimmune relative to any set in $\mathcal{M}$. For any $g \in \mathcal{X}(1)$, by Theorem 3.3.11, there are sets $\vec{G}$ witnessing the inequality $\mathsf{RT}^1_3 \not\leqslant_{\mathrm{soc}} (\mathsf{RT}^1_2)^*$. □

Liu [Liu23, Theorem 4.1] proved that $\mathsf{SRT}^2_3 \not\leqslant_c (\mathsf{SRT}^2_2)^*$. We strengthen his result by using Theorem 3.3.14 to prove that it holds even for non-stable instances of RT^2_2.

Theorem 3.5.20. $\mathsf{SRT}^2_3 \not\leqslant_c (\mathsf{RT}^2_2)^*$

Proof. By proposition 3.5.11 there exists a Δ^0_2 coloring $f : \mathbb{N} \to 3$ which is Γ-hyperimmune. Using Shoenfield's limit lemma, there is a stable computable function $h : [\mathbb{N}]^2 \to 3$ such that for every x, $\lim_y h(x, y) = f(x)$. Consider h as a computable instance of SRT^2_3. Fix any r-tuple of computable colorings $g_0, \dots, g_{r-1} : [\mathbb{N}]^2 \to 2$ for some $r \in \mathbb{N}$. It suffices to show the existence of g_i-homogeneous sets H_i for every $i < r$ such that $\bigoplus_{j<r} H_j$ does not compute any infinite h-homogeneous set.

By Corollary 3.5.18, there is a countable cross-constraint ideal $\mathcal{M} \models \mathsf{COH}$ for which f is Γ-hyperimmune relative to any set in $\mathcal{M}$. In particular, since $g_0, \dots, g_{r-1}$ are computable, they belong to $\mathcal{M}$. Moreover, by Lemma 3.5.10, f is hyperimmune relative to every element of $\mathcal{M}$. By Theorem 3.3.14, there exists g_i-homogeneous sets H_i for every $i < r$, such that $\bigoplus_{j<r} H_j$ does not compute any infinite f-homogeneous set. Since any h-homogeneous set is f-homogeneous, then $\bigoplus_{j<r} H_j$ does not compute any infinite h-homogeneous set. □

We now show that the previous theorem can be generalized to any $n > 2$, thus answering Question 12. Jockusch [Joc72b, Lemma 5.4] proved that for every computable coloring $f : [\mathbb{N}]^{n+1} \to k$, every PA degree relative to $\varnothing'$ computes an infinite pre-homogeneous set for f. Moreover, for $n \geqslant 2$, Hirschfeldt and Jocksuch [HJ16, Theorem 2.1] proved a reversal.

Theorem 3.5.21. *For every $n \geqslant 2$, $\mathsf{SRT}^n_3 \not\leqslant_c (\mathsf{RT}^n_2)^*$*

Proof. We prove by induction on $n \geqslant 2$ that for every set P, there exists a $\Delta^0_n(P)$ coloring $f : \mathbb{N} \to 3$ such that for every $r \geqslant 1$ every r-tuple of P-computable colorings $g_0, \dots, g_{r-1} : [\mathbb{N}]^n \to 2$, there are infinite $\vec{g}$-homogeneous sets $G_0, \dots, G_{r-1}$ such that $\vec{G} \oplus P$ does not compute any infinite f-homogeneous set.

The case $n = 2$ corresponds to a relativized form of Theorem 3.5.20. Now suppose the hypothesis holds for some $n \in \mathbb{N}$. Fix some set P, and let $Q \gg P'$ be such that $Q' \leqslant_T P''$. It exists by relativization of the low basis theorem (Jockusch and Soare [JS72b]).

By induction hypothesis relativized to Q, there exists a $\Delta^0_n(Q)$ (i.e. $\Delta^0_{n+1}(P)$) coloring $f : \mathbb{N} \to 3$ such that for every $r \geqslant 1$ every r-tuple of Q-computable colorings $g_0, \dots, g_{r-1} : [\mathbb{N}]^n \to 2$, there are infinite $\vec{g}$-homogeneous sets $G_0, \dots, G_{r-1}$ such that $\vec{G} \oplus Q$ does not compute any infinite f-homogeneous set.

Now, consider an r-tuple of P-computable colorings $h_0, \dots, h_{r-1} : [\mathbb{N}]^{n+1} \to 2$. By Jockusch [Joc72b, Lemma 5.4], Q computes infinite sets $C_0, \dots, C_{r-1} \subseteq \mathbb{N}$ prehomogeneous for $h_0, \dots, h_{r-1}$. For $s < r$, let $g_s : [\mathbb{N}]^n \to 2$ be the Q-computable coloring defined by $g_s(i_0, \dots, i_{n-1}) = h_s(x^s_{i_0}, \dots, x^s_{i_{n-1}}, y)$, where $C_s = \{x^s_0 < x^s_1 < \dots\}$ and $y \in C_s$ is any element bigger than $x^s_{i_{n-1}}$.

By choice of f, there are infinite $\vec{g}$-homogeneous sets $G_0, \dots, G_{r-1}$ such that $\vec{G} \oplus Q$ does not compute any infinite f-homogeneous set. In particular, letting $H_s = \{x^s_i : i \in G_s\}$, H_s is h_s-homogeneous, and since $\vec{H} \oplus P \leqslant_T \vec{G} \oplus Q$, then $\vec{H} \oplus P$ does not compute any infinite f-homogeneous set. This completes our induction.

Finally, by Shoenfield's limit lemma, for every $n \geqslant 2$, there exists a stable computable coloring $\widehat{f} : [\mathbb{N}]^n \to 3$ such that any infinite $\widehat{f}$-homogeneous set is f-homogeneous, where $f : \mathbb{N} \to 3$ is the function witnessed by the inductive proof. □

CHAPTER 4

THIN SET THEOREM AND OMNISCIENT REDUCTION

In this short chapter, we prove a separation result on the thin set theorem (a variation of Ramsey's theorem) and its stable counterpart. The proof uses forcing and is similar to the one established in [DPSW16].

The statements we are interested in are similar to RT^n_k, but the condition on the solution set is relaxed. Instead of homogeneity, it is only required that fewer than a fixed number of colors be used.

Statement 4.0.1 (Thin set theorem $\mathsf{RT}^n_{k,q}$)**.** For any any coloring $f : [\mathbb{N}]^n \to k$, there is an infinite set $H \subseteq \mathbb{N}$ such that $\operatorname{card}\big(f\big([H]^n\big)\big) \leqslant q$.

In particular $\mathsf{RT}^n_{k,k}$ is trivial, and $\mathsf{RT}^n_{k,1}$ is exactly RT^n_k. The statement $\forall k, \mathsf{RT}^n_{k,q}$ is denoted by $\mathsf{RT}^n_{<\infty,q}$. Moreover, if we consider only stable colorings, we denote the statement by $\mathsf{SRT}^n_{k,q}$. We can now state the theorem we wish to prove.

Theorem 4.0.2. *For all $q \geqslant 2$, we have* $\mathsf{RT}^1_{q+1,q} \not\leqslant_{soc} \mathsf{SRT}^2_{<\infty,q+1}$

Remark 4.0.3. To simplify notations in the proof, a binary string σ will often be identified with the set it represents, i.e. $\{x < |\sigma| : \sigma(x) = 1\}$.

Proof. We want to prove

$$\exists f : \mathbb{N} \to q+1, \forall g : [\mathbb{N}]^2 \to \ell \text{ stable}, \exists H \text{ solution of } g, \forall S \text{ solution of } f, H \not\geqslant_T S$$

Let $f : \mathbb{N} \to q+1$ be an $\mathcal{M}$-generic coloring for Cohen forcing, where $\mathcal{M}$ is a countable transitive model of Z_2, and let $g : [\mathbb{N}]^2 \to \ell$ be a stable coloring. We are going to construct H by forcing, but first we use the following lemma to define two sets $B \in \mathcal{M}$ and $J \subseteq \ell$ that come into play in the forcing conditions we are going to use. The idea behind this lemma is to carefully tailor B and J, so that we always find suitable forcing extensions later. Its usage will become clearer once we reach cases 2 and 3 of the construction.

Lemma 4.0.4 (target colors). *There is an infinite set $B \in \mathcal{M}$, and a set $J \coloneqq \{j_0, \dots, j_q\} \subseteq \ell$ such that, for any $i \leqslant q$, for any infinite $C \subseteq B$ in $\mathcal{M}$ and for any injection $h : C \to \mathbb{N}$ in $\mathcal{M}$, we have*

$$\exists^\infty x \in C, f(h(x)) = i \wedge \lim_y g(x, y) = j_i$$

Proof. **Construction.** For every $i \leqslant q$, every $F \subseteq \ell$ and every infinite set A, consider the class $\mathcal{D}^A_{i,F}$ whose elements are infinite sets $B \subseteq A$ in $\mathcal{M}$ such that, for all infinite subset $C \subseteq B$ in $\mathcal{M}$ and for all $h : C \to \mathbb{N}$ in $\mathcal{M}$, we have $\exists^\infty x \in C, f(h(x)) = i \wedge \lim_y g(x, y) \notin F$.

Note that $\mathcal{D}^A_{i,\varnothing} \neq \varnothing$ for any $i \leqslant q$ and any infinite set A. Otherwise, for all infinite $B \subseteq A$ in $\mathcal{M}$, there is $C \subseteq B$ in $\mathcal{M}$ and $h : C \to \mathbb{N}$ in $\mathcal{M}$ such that $\forall^\infty x \in C, (f(h(x)) = i \implies \lim_y g(x, y) \in \varnothing)$. And since f is $\mathcal{M}$-generic, there are infinitely many $x \in C$ such that $f(h(x)) = i$, leading to a contradiction. Moreover note that $\mathcal{D}^A_{i,\ell} = \varnothing$ for any $i \leqslant q$ and any infinite set A. Otherwise, there would be elements whose limit color for g is not in ℓ, a contradiction.

We now construct B by induction. First define $B_0 \coloneqq \mathbb{N}$ and $F_0 \coloneqq \varnothing$. Suppose B_i and F_i have been constructed for some $i \leqslant q$. Let $F_{i+1} \subseteq \ell$ be maximal for the inclusion such that $\mathcal{D}^{B_i}_{i,F_{i+1}} \neq \varnothing$. Then B_{i+1} is chosen as an element of this class. Afterwards, define $B \coloneqq B_{q+1}$ and $J \coloneqq \{j_i : i \leqslant q\}$ where j_i is some color in $\ell - F_{i+1}$ (which is not empty since $\mathcal{D}^{B_i}_{i,\ell} = \varnothing$).

Verification. By contradiction suppose there is $i \leqslant q$, $C \subseteq B$ in $\mathcal{M}$ and $h : C \to \mathbb{N}$ in $\mathcal{M}$, such that $\forall^\infty x \in C, \big(f(h(x)) = i \implies \lim_y g(x, y) \neq j_i\big)$. Then F_{i+1} was not maximal for the inclusion during the construction, because $C \in \mathcal{D}^{B_i}_{i,G}$ where $G \coloneqq F_{i+1} \cup \{j_i\}$. □

We can now define our forcing conditions to be Mathias conditions (σ, X) such that

- $g\big([\sigma]^2\big) \subseteq J$
- $\forall x \in \sigma, \forall y \in X, g(x, y) \in J$
- $X \subseteq B \in \mathcal{M}$

For any c.e. functional W and any color $i \leqslant q$, we define the requirement

$$\begin{aligned}\mathcal{R}_{W,i} \coloneqq W^H &\text{ is not an infinite set}\\ &\text{or there is } w \in W^H \text{ such that } f(w) = i\end{aligned}$$

We are going to construct a sequence of conditions, starting with $(\varepsilon, \mathbb{N})$ and such that, for any functional W and any color $i \leqslant q$, there is a condition forcing $\mathcal{R}_{W,i}$. In other words, if W^H is an infinite set, then it uses more than q colors for f.

The search for extensions will be guided by trees. For each n we define the tree $T_n \subseteq X^{<\mathbb{N}}$, such that $\varepsilon \in T_n$, and $\alpha \in T_n$ iff $\alpha \in X^{<\mathbb{N}}$ is strictly increasing and

$$\forall \rho \subseteq \operatorname{ran}\left(\alpha^{\#}\right), W^{\sigma \cup \rho} \subseteq [\![0, n[\![$$

Where the operator $\cdot^{\#}$ takes a finite sequence and returns it without its last element. Note that $\forall n < m, T_n \subseteq T_m$, by definition. We now present a few lemmas on these trees to better understand them.

Lemma 4.0.5 (non-terminal node)**.** *Let α be a node of T_n. If it is not terminal, then it has infinitely many successors.*

Proof. Let α be a non-terminal node of T_n. By definition, there exists some $x \in X$ such that $\alpha \cdot x$ is strictly increasing and $\forall \rho \subseteq \operatorname{ran}\left((\alpha \cdot x)^{\#}\right), W^{\sigma \cup \rho} \subseteq [\![0, n[\![$. But $(\alpha \cdot x)^{\#} = \alpha$, so the above formula does not depend on x. Hence, for any $y \in X$ such that $\alpha \cdot y$ is strictly increasing, y is a successor of α. And since X is infinite, there are infinitely many such elements. □

Lemma 4.0.6 (terminal node)**.** *Let α be a node of the tree T_n. If it is terminal, then there exists $\rho \subseteq \operatorname{ran}\alpha$ and $w \geqslant n$ such that $w \in W^{\sigma \cup \rho}$.*

Proof. If α has no successor, then for any $x \in X$ such that $\alpha \cdot x$ is strictly increasing, we have $\exists \rho \subseteq \operatorname{ran}\left((\alpha \cdot x)^{\#}\right), W^{\sigma \cup \rho} \not\subseteq [\![0, n[\![$, i.e. $\exists \rho \subseteq \operatorname{ran}\alpha, \exists w \geqslant n, w \in W^{\sigma \cup \rho}$. □

To refer to this lemma we may say "the ρ and w associated to the (terminal) node α". We now prove the main claim of the theorem.

Claim 4.0.7. *For any condition (σ, X), any functional W and any color $i \leqslant q$, there is an extension of (σ, X) which forces $\mathcal{R}_{W,i}$.*

Proof. To extend (σ, X) we distinguish several cases.

Case 1. If there is n such that T_n is not well-founded, then we can force W^H to be finite. Indeed, let P be an infinite path of this tree, and let Y be its range. We have that $Y \subseteq X$ and that $\forall \rho \subseteq Y, W^{\sigma \cup \rho} \subseteq [\![0, n[\![$. Thus by defining the new extension (σ, Y), we have the desired result.

We now assume all the trees T_n are well-founded, and we are going to force the "or" case of the requirement. More precisely, given (σ, X) and $i \leqslant q$, we try to find a terminal node γ in some T_n, whose w associated by the terminal node lemma is such that $f(w) = i$. By the lemma there is also $\rho \subseteq \operatorname{ran} \gamma$ such that $w \in W^{\sigma \cup \rho}$. So we can extend (σ, X) into $\big(\sigma \cup \rho, X \cap [\![m, \infty[\![\big)$, where m is such that $\forall x \in \sigma \cup \rho, \forall y \in X, (y > m \implies g(x, y) \in J)$. To help us find the desired terminal node we label the trees:

Definition 4.0.8 (labelling)**.** The nodes of each well-founded T_n are labelled recursively, by an integer or the symbol ∞, we begin with terminal nodes:

- If the node α is terminal, then we know there exist w and $\rho \subseteq \operatorname{ran} \alpha$ such that $w \in W^{\sigma \cup \rho}$, we thus label α with the smallest w of this kind.
- If α is non-terminal and if all its successors have a label, then two possibilities arise. Either one of the labels appears infinitely many times amongst the successors, we thereby find the smallest such label w, and we assign it to α. Or there is no such label, and we thereby assign the label ∞ to α.

Informally, if α has a finite label w, then it can be extended to a terminal node whose associated value is w. The label is ∞ if there are infinitely many different labels α could lead to. Before proceeding we prune all the trees T_n, so they are easier to manipulate. This procedure does not change the label associated to a node.

Definition 4.0.9. To each T_n we associate T_n^L, defined recursively from its root.

- The root of T_n^L is ε.
- If α is in T_n^L and is labeled w, then all its successors with the same label are also in T_n^L.
- If α is in T_n^L and has label ∞, and if it has infinitely many successors with the same label, then all these successors are in T_n^L.
- If α has label ∞ but only finitely many successors with this label, then there are infinitely many different labels $w_0 \leqslant w_1 \leqslant \dots$ amongst its successors. (because the label of α is ∞, so it is non-terminal and it cannot have infinitely many successors with a finite label w). For each $i \in \mathbb{N}$, $\alpha \cdot x_i$ is in T_n^L, where x_i is the smallest integer such that $\alpha \cdot x_i$ has label w_i.

The structure of T_n^L is basically the same as T_n. The terminal nodes are the same, and a non-terminal node in T_n^L still has infinitely many successors. But now we also have the following lemma.

Lemma 4.0.10 (labelling)**.** *For any n, let $\alpha \in T_n^L$ with a finite label w. In the sub-tree whose root is α, all the nodes have the label w.*

Case 2. We suppose here that every tree T_n^L has a finite label, noted w_n, at its root. The set $V \coloneqq \{w_n : n \in \mathbb{N}\}$ is an infinite set of $\mathcal{M}$, because $w_n \geqslant n$ by the terminal node lemma. By genericity of f, and since V is an infinite set of $\mathcal{M}$, we have that, for any color $i \leqslant q$, there are infinitely many elements of V which are colored i by f. So we can find n such that $f(w_n) = i$. Recall that all the nodes in T_n^L have label w_n.

We look for a terminal node $\gamma \in T_n^L$ and some m such that $\big(\gamma, X \cap [\![m, \infty[\![\big)$ is a valid condition, then, by the terminal node lemma, there is $\rho \subseteq \operatorname{ran} \gamma$ such that $w_n \in W^{\sigma \cup \rho}$ and we have that $\big(\sigma \cup \rho, X \cap [\![m, \infty[\![\big)$ is a valid condition that forces $\mathcal{R}_{W,i}$ since $f(w_n) = i$. So we construct a sequence $\gamma_0 \prec \gamma_1 \prec \dots$ of nodes of T_n^L such that, for all s, $\big(\gamma_s, X \cap [\![m_s, \infty[\![\big)$ is a condition and some $m_s \in \mathbb{N}$. Since the tree is well-founded, this sequence must be finite, and γ will be chosen as its greatest element.

We construct the sequence by induction, starting with $\gamma_0 \coloneqq \varepsilon$. Suppose that γ_s is constructed. If it is terminal, we are done. Otherwise consider the row under γ_s, noted $A \coloneqq \{x \in X : \gamma_s \cdot x \in T_n^L\} \in \mathcal{M}$. Note that $A \subseteq B$ since $X \subseteq B$.

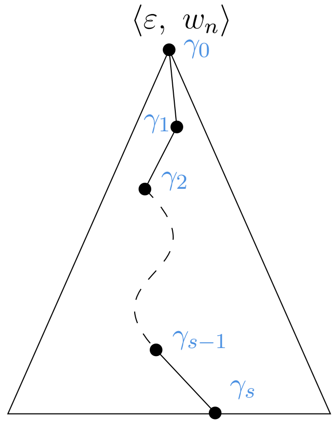

Figure 4.1: A representation of the construction of γ_s.

According to the target colors lemma (with the identity function as h), there is some $x \in A \cap [\![m_s, \infty[\![$ such that $\lim_y g(x, y) \in J$. Moreover, since $\gamma_s \cdot x \in T_n^L$, its label is w_n and verifies $f(w_n) = i$. In which case define $\gamma_{s+1} \coloneqq \gamma_s \cdot x$, and let m_{s+1} be the smallest integer verifying $\forall z \in \gamma_{s+1}, \forall y \in X, (y > m_{s+1} \implies g(z, y) \in J)$.

Case 3. Finally, we suppose there is some n such that the root of T_n^L is labelled ∞. We begin the same construction as in Case 2 to find γ, but we stop when a node γ_s has label ∞ while all its successors have a finite label. This is bound to happen, since the root of the tree has label ∞ and terminal nodes do not. We then consider the row under γ_s, noted $A \coloneqq \{x \in X : \gamma_s \cdot x \in T_n^L\} \in \mathcal{M}$, and $h \in \mathcal{M}$ the function which, to an element $x \in A$, associates the label of $\gamma_s \cdot x$. It is an injection due to the pruning we made on trees. Note that $A \subseteq B$ since $X \subseteq B$. By the target colors lemma, there is some $x \in A \cap [\![m_s, \infty[\![$ such that $f(h(x)) = i$ and $\lim_y g(x, y) \in J$. In which case we define $\gamma_{s+1} \coloneqq \gamma_s \cdot x$ and then continue the construction, just like in the previous case, to obtain the terminal node γ whose label $w \coloneqq h(x)$ verifies $f(w) = i$, and such that $(\gamma, X \cap [\![m, \infty[\![)$ for some $m \in \mathbb{N}$.

Thus, we use the terminal node lemma to find $\rho \subseteq \operatorname{ran} \gamma$ such that $w \in W^{\sigma \cup \rho}$, and so the extension $\big(\sigma \cup \rho, X \cap [\![m, \infty[\![\big)$ forces $\mathcal{R}_{W,i}$. □

□

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