Computational analysis of Ramsey-type theorems

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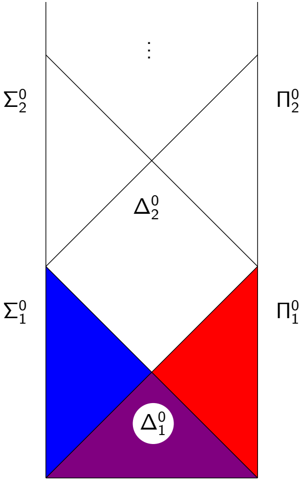
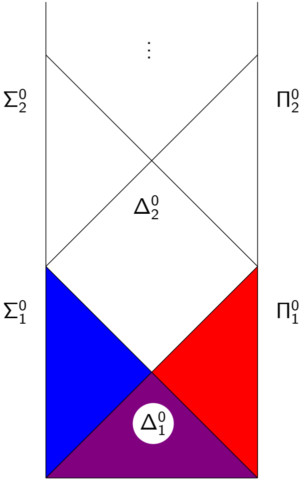
The arithmetic hierarchy



$$\Sigma^0_1 : \{x : \exists y, (x,y) \in R\}$$
$$\Pi^0_1 : \{x : \forall y, (x,y) \in R\}$$
$$\Sigma^0_2 : \{x : \exists y, \forall z, (x,y,z) \in R\}$$
$$\Pi^0_2 : \{x : \forall y, \exists z, (x,y,z) \in R\}$$
$$\vdots$$

$$\Delta^0_n := \Sigma^0_n \cap \Pi^0_n$$

The arithmetic hierarchy



Theorem (Post)

Let $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$

$$A \text{ is } \emptyset^{(n)}\text{-computable} \iff A \text{ is } \Delta^0_{n+1}$$
$$A \text{ is } \emptyset^{(n)}\text{-c.e.} \iff A \text{ is } \Sigma^0_{n+1}$$

$\Delta^0_1 \equiv$ computable

$\Sigma^0_1 \equiv$ computably enumerable

$\Delta^0_2 \equiv \emptyset'$-computable

What are reverse mathematics ?

A branch of logic created in 1974 by Harvey Friedman

Goals:

- Find the simplest axioms required to prove a given theorem (hence the name)
- Provide an adequate framework to investigate the computational content of theorems
- Search for new proofs

The framework of reverse mathematics

- Second order arithmetic, with structures of the form $\langle N, S\rangle$
- An ω-**model** is of the form $\langle \omega, S\rangle$ where ω is the set of standard integers
- We must be able to encode the objects we are working with, for example continuous functions from $\mathbb{R}$ to $\mathbb{R}$
- Subsystems above a base theory called RCA_0

Comprehend RCA_0

- Corresponds to computable mathematics
- Its typical model is $\langle \omega, \mathrm{COMP} \rangle$ where COMP is the class of computable sets
- We now have a more faithful definition of the implication between two theorems $\mathsf{RCA}_0 \vdash \mathsf{P} \Rightarrow \mathsf{Q}$.

RCA_0 Recursive Comprehension Axiom

- Robinson's arithmetic
- Comprehension scheme for Δ_1^0 formula

$$(\forall x, (\varphi(x) \Leftrightarrow \psi(x))) \Rightarrow \exists X, \forall y, (y \in X \Leftrightarrow \varphi(y))$$

- Induction scheme for Σ_1^0 formula

$$\varphi(0) \wedge \Big((\forall x, (\varphi(x) \Rightarrow \varphi(x+1))) \Rightarrow \forall x, \varphi(x) \Big)$$

Turing ideals

A **Turing ideal** is a class $S \subseteq 2^{\mathbb{N}}$ that is closed for Turing reduction and join.

Theorem (Friedman)

Let $\mathcal{M} := \langle \omega, S \rangle$, we have

$$\mathcal{M} \models \mathsf{RCA}_0 \iff S \text{ is a Turing ideal}$$

$\langle \omega, \mathrm{COMP} \rangle$ is the smallest model of RCA_0 for the inclusion

The Big Five

Π^1_1-CA

$\downarrow$

ATR_0

$\downarrow$

ACA_0

$\downarrow$

WKL_0

$\downarrow$

RCA_0

Weak König's Lemma - WKL

Every infinite binary tree has an infinite path.

$$WKL_0 := RCA_0 \cup \{WKL\}$$

$$ACA_0 := RCA_0 \cup \text{Arithmetical comprehension axioms}$$

$$= RCA_0 \cup \{\text{``Every set has a Turing jump''}\}$$

The Big Five Phenomenon

Π^1_1-CA

↓

ATR_0

↓

ACA_0

↓

WKL_0

↓

RCA_0

Empirical result: a "classical" theorem is either provable in RCA_0 or equivalent (modulo RCA_0) to one of the four other subsystems.

Questions

Why is this the case? Are there natural statements that escape this phenomenon?

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What is Ramsey's theorem?

Figure: Frank P. Ramsey

What is Ramsey's theorem? (informally)

$$
\begin{array}{rcl}
(0,1) \mapsto {\color{red}\blacksquare} & & (0,{\color{gray}1}) \mapsto {\color{gray}\blacksquare} \\
(0,2) \mapsto {\color{red}\blacksquare} & & (0,2) \mapsto {\color{red}\blacksquare} \\
(1,2) \mapsto {\color{blue}\blacksquare} & & ({\color{gray}1},2) \mapsto {\color{gray}\blacksquare} \\
(2,100) \mapsto {\color{red}\blacksquare} & & (2,100) \mapsto {\color{red}\blacksquare} \\
(2,109) \mapsto {\color{blue}\blacksquare} & \rightsquigarrow & (2,{\color{gray}109}) \mapsto {\color{gray}\blacksquare} \\
(100,108) \mapsto {\color{blue}\blacksquare} & & (100,{\color{gray}108}) \mapsto {\color{gray}\blacksquare} \\
(100,109) \mapsto {\color{red}\blacksquare} & & (100,{\color{gray}109}) \mapsto {\color{gray}\blacksquare} \\
(100,110) \mapsto {\color{red}\blacksquare} & & (100,110) \mapsto {\color{red}\blacksquare} \\
\vdots & & \vdots
\end{array}
$$

What is Ramsey's Theorem? (formally)

For every $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, we define the set of n-**tuples**

$$[X]^n := \{F \subseteq X : |F| = n\}$$

For every $k \in \mathbb{N}$, a k-**coloring** of $[X]^n$ is a function from $[X]^n$ to $\{0, \dots, k-1\}$.

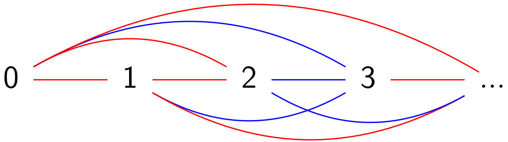
A set H is f-**homogeneous** if there is a color $i < k$ such that $f\big([H]^n\big) = i$.

Ramsey's Theorem

Ramsey's Theorem - RT^n_k

For every k-coloring f of $[\mathbb{N}]^n$, there exists an infinite f-homogeneous set.

For $n = 1$, it is the infinite pigeonhole principle. Example for $n = 2$ and $k = 2$:



A first property

Over RCA_0, it is equivalent to consider colorings of $[\mathbb{N}]^n$ instead of $[X]^n$, for any infinite set $X \subseteq \mathbb{N}$.

Usage: first $f_0 \xrightarrow{\mathsf{RT}} H_0$, then $f_1 {\restriction}_{H_0} \xrightarrow{\mathsf{RT}} H_1$

Theorem (Jockusch 1972)

For any n and any $k \geqslant 2$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \implies \mathsf{RT}^n_{k+1}$

Given $f : [\mathbb{N}]^n \to k+1$, consider

$$\widehat{f} : [\mathbb{N}]^n \to k$$

$$\overline{x} \mapsto \begin{cases} k-1 & \text{when } f(\overline{x}) = k \\ f(\overline{x}) & \text{otherwise} \end{cases}$$

Hierarchy around Ramsey's Theorem

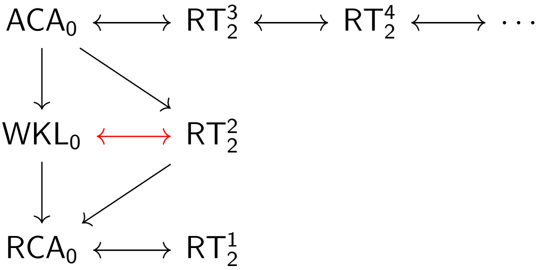


Figure: Ramsey's Theorem in the Big Five hierarchy. A red arrow means the implication does not hold.

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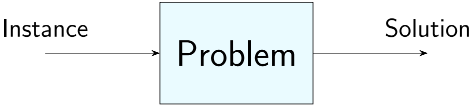
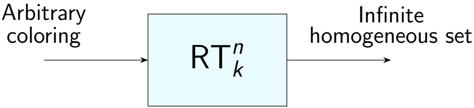
Theorems as problems

Many theorems are of the form

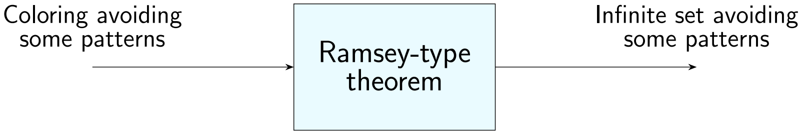
$$\forall X(\Phi(X) \Rightarrow \exists Y, \Psi(X, Y))$$



Examples of problems



Ramsey-type theorems

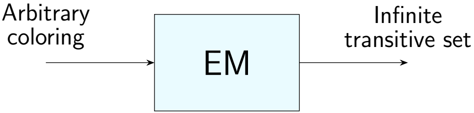


Examples of Ramsey-type theorems

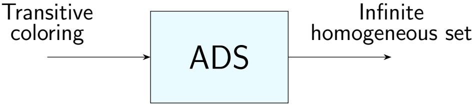
Consider **transitivity** (for $n = 2$ and $k = 2$)

$$\forall i < 2, f(x,y) = f(y,z) = i \Rightarrow f(x,z) = i$$

Erdős-Moser:



Ascending Descending Sequence:



Studying forbidden patterns for RT^2_2

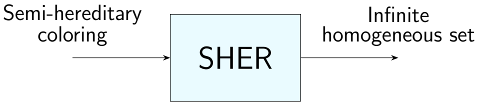
We consider the following forbidden patterns for 3 variables and 2 colors

	$f(x,y)$	$f(y,z)$	$f(x,z)$
Transitivity	blue	blue	red
Ascendancy	blue	red	blue
Heredity	red	blue	blue

	Input	**Output**
Transitivity	$< \mathsf{RT}^2_2$ ($\Leftrightarrow$ CAC)	$< \mathsf{RT}^2_2$ ($\leqslant$ EM)
Ascendancy	$< \mathsf{RT}^2_2$ ($\Leftrightarrow \mathsf{B}\Sigma^0_2$)	$\Leftrightarrow \mathsf{RT}^2_2$
Heredity	$< \mathsf{RT}^2_2$ ($=$ SHER)	$\Leftrightarrow \mathsf{RT}^2_2$

SHER

SHER was first studied by Dorais:



Theorem

In RCA_0, the following are equivalent:

- SHER
- CAC for trees
- $\mathsf{TAC} + \mathsf{B}\Sigma^0_2$

Statements about trees

CAC for (c.e.) trees

Every infinite (c.e.) subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path or an infinite antichain.

Tree AntiChain Principle - TAC (Conidis)

Every subtree of $2^{<\mathbb{N}}$ that is c.e., infinite, and completely branching, i.e., $\forall i < 2, \forall \sigma \in T, (\sigma \cdot i \in T \Rightarrow \sigma \cdot (1 - i))$, has an infinite antichain.

The Complete Equivalence

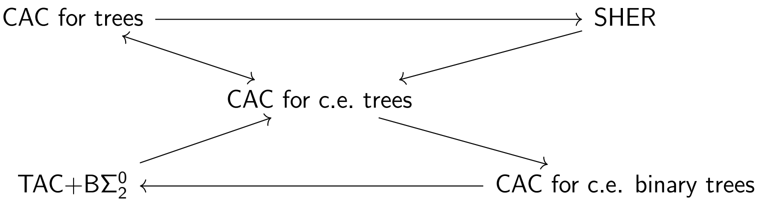


Figure: Implications between the different equivalent versions of CAC for trees

Some results on TAC

Theorem

For every uniformly Δ^0_2 sequence $(A_n)_{n\in\mathbb{N}}$ of infinite Δ^0_2 sets, there exists a computable instance of TAC *such that none of the A_n is a solution.*

Corollary

For every low set P, there exists a computable instance of TAC without a P-computable solution.

CAC for trees in the hierarchy

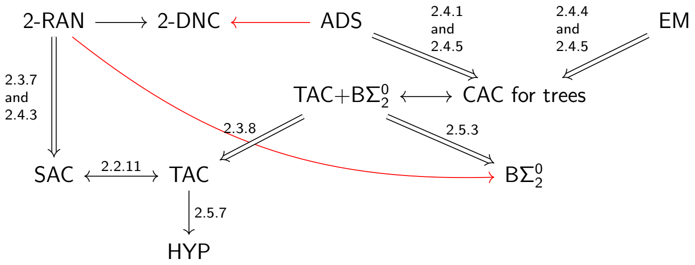


Figure: The different implications between CAC for trees and other known statements. A double arrow means that there is a strict implication, a red arrow that there is no implication.

How are these implications proved?

$$RCA_0 \vdash TAC + B\Sigma^0_2 \Rightarrow \text{CAC for trees}$$

Proof.

We have $T \subseteq \mathbb{N}^{<\mathbb{N}}$ an infinite c.e. tree. We want an infinite antichain or chain.

1. T has a node with infinitely many children
2. T has a finite number of branching nodes
3. Otherwise, we construct a completely branching infinite c.e. tree S and a function $f : S \to T$ such that:

$$\forall \sigma, \nu \in S, \sigma | \nu \Rightarrow f(\sigma) | f(\nu)$$

□

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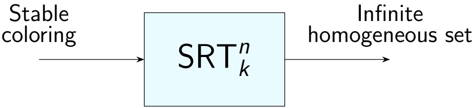
Stable Ramsey's theorem



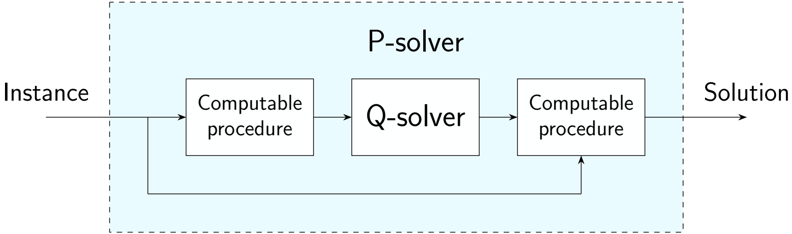
Stability for $f : [\mathbb{N}]^{n+1} \to k$

$$\forall \vec{x} \in [\mathbb{N}]^n, \lim_y f(\vec{x}, y) \text{ exists}$$

k	1	2	3	...	s	$s+1$	$s+2$	...
$f(x, x+k)$	■	■	■	...	■	■	■	...

Computable reduction

$$P \leqslant_c Q$$



RT^n_k and computable reduction

The number of colors matter now

Theorem (Patey)

For all $n \geqslant 2$ and all $k > \ell \geqslant 2$, we have $\mathrm{SRT}^n_k \not\leqslant_c \mathrm{RT}^n_\ell$

In particular $\mathrm{RT}^n_{k+1} \not\leqslant_c \mathrm{RT}^n_k$

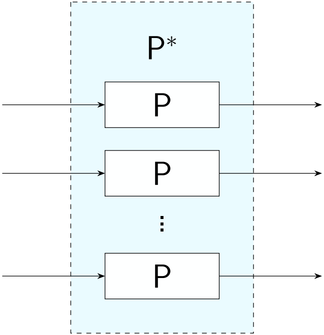
P $\not\leqslant_c$ Q means there is an instance I of P whose solutions are complex enough, so that, for any I-computable instance $\widehat{I}$ of Q, there is a Q-solution $\widehat{S}$ of $\widehat{I}$ such that $I \oplus \widehat{S}$ does not compute any P-solution of I.

To create such an instance we use forcing.

Product of problems

Question

Is it possible to solve RT^n_{k+1} by using multiple instances of RT^n_k chosen *simultaneously*?



Let P^* be the **star** of the problem P.

Does $\mathsf{RT}^n_{k+1} \leqslant_c (\mathsf{RT}^n_k)^*$ hold?

Answering the question

Through some technical new notions and combinatorial arguments, Liu proved the following:

Theorem (Liu)

$\mathsf{SRT}^2_3 \not\leqslant_c (\mathsf{SRT}^2_2)^*$

By exploiting his method, we have shown:

Theorem

For all $n \geqslant 2$, $\mathsf{SRT}^n_3 \not\leqslant_c (\mathsf{RT}^n_2)^*$

This proof relies on a statement called COH.

Cohesiveness

A set A is **almost included** in a set B, noted $A \subseteq^* B$, if $\forall^\infty x \in A, x \in B$

COH

For every infinite sequence of sets $\vec{R}$, there exists an infinite set C that is $\vec{R}$-**cohesive**, i.e., for every $i \in \mathbb{N}$, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$

COH turns arbitrary colorings into stable colorings, by considering the sequence $R_{x,i} := \{y \in \mathbb{N} : f(x,y) = i\}$ (the limit coloring $g : x \mapsto \lim f(x, -)$ is $(C \oplus f)'$-computable).

Preservation

P **preserves** $\mathcal{W}$: if $Z \subseteq \mathbb{N}$ verifies a property $\mathcal{W}$, then, for any Z-computable instance X, there is a solution Y such that $Z \oplus Y$ also verifies $\mathcal{W}$.

Proposition

If P preserves a weakness property $\mathcal{W}$. Then for all $X \in \mathcal{W}$, there is a model $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{P}$ whose second-order part is a class $S \subseteq \mathcal{W}$ such that $X \in S$.

Proposition

If a problem Q preserves a weakness property $\mathcal{W}$, but a problem P does not, then $\mathsf{RCA}_0 + \mathsf{Q} \nvdash \mathsf{P}$.

Sketch of proof

First step: There exists a Δ^0_2 coloring $f : \mathbb{N} \to 3$ that is Γ-hyperimmune.

Second step: COH and CC preserve Γ-hyperimmunity.

Third step: There a model $\mathcal{M}$ of COH and CC such that f is Γ-hyperimmune relative to all the elements of $\mathcal{M}$.

Fourth step: Since $\mathcal{M} \models$ COH, we can turn the instances of $(\mathsf{RT}^2_2)^*$ into stable colorings and then rely on Liu's proof of $\mathsf{SRT}^2_3 \not\leq_c (\mathsf{SRT}^2_2)^*$.

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Conclusion

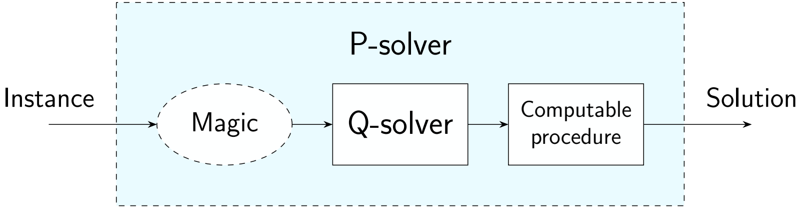
Julien Cervelle, William Gaudelier, Ludovic Patey
The Reverse Mathematics of CAC for trees (2022)
The Journal of Symbolic Logic

Julien Cervelle, William Gaudelier, Ludovic Levy Patey
Cross-constraint basis theorems and products of partitions (2024)
submitted

Thank you for your attention

Result on thin set theorem

Strong omniscient computable reduction :



Theorem

For any $q \geqslant 2$, $\mathsf{RT}^1_{q+1,q} \not\leq_{soc} \mathsf{SRT}^2_{<\infty,q+1}$

Theorem (Dzhafarov et al.)

For all $k > \ell$, $\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$

Bounding scheme

$$\forall a, \big((\forall x < a, \exists y, \varphi(x,y)) \Rightarrow \exists b, \forall x < a, \exists y < b, \varphi(x,y)\big)$$

Theorem (Kirby, Paris)

For all n, $\mathsf{I}\Sigma^0_{n+1} \Rightarrow \mathsf{B}\Sigma^0_{n+1} \Rightarrow \mathsf{I}\Sigma^0_n$

Theorem (Hirst)

$\mathsf{RCA}_0 \vdash \mathsf{B}\Sigma^0_2 \Leftrightarrow \forall k, \mathsf{RT}^1_k$

Hyperimmunity

Definition - Hyperimmunity

A function $f : \mathbb{N} \to k$ is **hyperimmune** relative to $D \subseteq \mathbb{N}$ if, for every D-computable sequence of mutually disjoint finite k-tuples $((F_{n,0}, \dots, F_{n,k-1}))_{n\in\mathbb{N}}$ such that $\bigcup_{j<k} F_{n,j} > n$, there exists $m \in \mathbb{N}$ such that

$$\forall j < k, F_{m,j} \subseteq f^{-1}(j)$$

Mathias forcing

A **Mathias condition** is a pair (σ, X) where

- σ is a binary string (identified with a finite set)
- $X \in 2^{\mathbb{N}}$ is an infinite set, called **reservoir**
- $X \cap \{0, \dots |\sigma| - 1\} = \emptyset$

A Mathias condition (τ, Y) extends another (σ, X), if $Y \subseteq X$ and $\tau \succcurlyeq \sigma$ where $\tau - \sigma \subset X$.

The interpretation of a Mathias condition is given by $\left[(\sigma, X)\right] := \{Z \in [\sigma] : Z \subseteq \sigma \cup X\}$.

Other forcing (3.3.11)

A variant of Mathias forcing with conditions of the form $((\vec{F}_\alpha)_{\alpha\in 2^r}, \vec{A})$ where

- $\vec{F}_\alpha$ is an r-tuple of finite sets g-homogeneous for the colors α, i.e. $\forall s < r, F_{\alpha,s} \subseteq g_s^{-1}(\alpha(s))$
- $\vec{A}$ is an r-tuple of infinite sets in $\mathcal{M}$ such that $\forall \alpha \in 2^r, \forall s < r, \min A_s > \max F_{\alpha,s}$

A condition $((\vec{E}_\alpha)_{\alpha\in 2^r}, \vec{B})$ *extends* another $((\vec{F}_\alpha)_{\alpha\in 2^r}, \vec{A})$ if, for every $\alpha \in 2^r$ and every $s < r$, we have $E_{\alpha,s} \supseteq F_{\alpha,s}$, $B_s \subseteq A_s$, and $E_{\alpha,s} - F_{\alpha,s} \subseteq A_s$.

Cross-trees

$$\mathcal{X}_{<\mathbb{N}} := 3^{<\mathbb{N}} \times (2^{<\mathbb{N}})^r$$

Definition

A **cross-tree** is a set $T \subseteq \mathcal{X}_{<\mathbb{N}}$ which is downward-closed for the prefix relation $\preccurlyeq$, and such that $\forall(\rho, \sigma) \in T, \forall i<j<r, |\sigma_i| = |\sigma_j|$ and $|\sigma| \leqslant |\rho|$.

Definition - Proposition

A class $\mathcal{P} \subseteq \mathcal{X}$ is *left-full* below $(\rho, \sigma) \in \mathcal{X}_{<\mathbb{N}}$ if

$$\forall X \in [\rho], \exists Y \in [\sigma], (X, Y) \in \mathcal{P}$$

Equivalently

$$\forall \mu \succcurlyeq \rho, \exists \tau \succcurlyeq \sigma, |\tau| = |\mu| \text{ and } (\mu, \tau) \in T$$

Cross-constraint

Definition

Let X be an infinite set. A pair of instances (f, g) of RT^1_k is *finitely compatible on X* if for all color $i < k$ the set $X \cap f^{-1}(i) \cap g^{-1}(i)$ is finite. Whenever $X = \mathbb{N}$, we simply say that (f, g) is finitely compatible. Also, note that the negation of "finitely compatible" is "infinitely compatible"

CC

For any left-full cross-tree $T \subseteq \mathcal{X}_{<\mathbb{N}}$, there is a pair of paths $(X^i, Y^i)_{i<2}$ such that (X^0, X^1) is finitely compatible, and for all $s < r$, (Y^0_s, Y^1_s) is infinitely compatible.

Preservation of weakness properties

A **weakness property** is a class $\mathcal{W} \subseteq 2^{\mathbb{N}}$ that is downward-closed for Turing reduction.

Example $\mathcal{W}_f := \{X \subseteq \mathbb{N} : f \text{ is hyperimmune relative to } X\}$

A problem P **preserves** a weakness property $\mathcal{W}$ if, for all $Z \in \mathcal{W}$, any Z-computable instance of P admits a solution Y such that $Z \oplus Y \in \mathcal{W}$.

COH preserves hyperimmunity.